11.3 Solving Radical Equations

Essential Question: How can you solve equations involving square roots and cube roots?

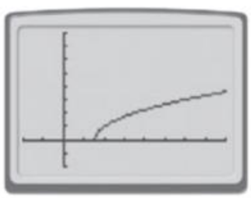
NOTE: Many students might not have a graphing calculator, so you can download one from the internet at the APP store for free.

Explore Investigating Solutions of Square Root Equations

When solving quadratic equations, you have learned that the number of real solutions depends upon the values in the equation, with different equations having 0, 1, or 2 real solutions. How many real solutions does a square root equation have? In the Explore, you will investigate graphically the numbers of real solutions for different square root equations.

(A) Remember that you can graph the two sides of an equation as separate functions to find solutions of the equation: a solution is any x-value where the two graphs intersect.

The graph of $y = \sqrt{x-3}$ is shown on a calculator window of $-4 \leq x \leq 16$ and $-2 \leq y \leq 8$. Reproduce the graph on your calculator. Then add the graph of $y = 2$.



How many solutions does the equation $\sqrt{x-3} = 2$ have? ______ How do you know?

____

On your calculator, replace the graph of $y = 2$ with the graph of $y = -1$.

How many solutions does the equation $\sqrt{x-3} = -1$ have? ______ How do you know?

____

How many solutions does the equation $\sqrt{x-3} = 2$ have? **one** How do you know?

The graphs intersect at one point.

On your calculator, replace the graph of $y = 2$ with the graph of $y = -1$.

How many solutions does the equation $\sqrt{x-3} = -1$ have? **zero** How do you know?

The graphs never intersect.

(B) Graph $y = \sqrt{x-3} + 2$ on your calculator (you can use the same viewing window as in Step A).

Add the graph of $y = 3$ to the graph of $y = \sqrt{x-3} + 2$.

How many solutions does $\sqrt{x-3} + 2 = 3$ have? ______

Replace the graph of $y = 3$ with the graph of $y = 1$.

How many solutions does $\sqrt{x-3} + 2 = 1$ have? ______

(B) Graph $y = \sqrt{x-3} + 2$ on your calculator (you can use the same viewing window as in Step A).

Add the graph of $y = 3$ to the graph of $y = \sqrt{x-3} + 2$.

How many solutions does $\sqrt{x-3} + 2 = 3$ have? one

Replace the graph of $y = 3$ with the graph of $y = 1$.

How many solutions does $\sqrt{x-3} + 2 = 1$ have? zero

Essential Question: How can you solve equations involving square roots and cube roots?

First, combine terms to simplify, if possible. If there is one radical expression, isolate it on one side of the equation. Then, to solve the equation, square both sides when a square root is involved, and cube both sides when a cube root is involved. Finally, solve the resulting equation for the variable, remembering to check for extraneous solutions

(C) Graph both sides of $\sqrt{4x - 4} = x + 1$ as separate functions on your calculator.

How many solutions does $\sqrt{4x - 4} = x + 1$ have? ______

Replace the graph of $y = x + 1$ with the graph of $y = \frac{1}{2}x$.

How many solutions does $\sqrt{4x - 4} = \frac{1}{2}x$ have? ______

Replace the graph of $y = \frac{1}{2}x$ with the graph of $y = 2x - 5$.

How many solutions does $\sqrt{4x - 4} = 2x - 5$ have? ______

(C) Graph both sides of $\sqrt{4x - 4} = x + 1$ as separate functions on your calculator.

How many solutions does $\sqrt{4x - 4} = x + 1$ have? zero

Replace the graph of $y = x + 1$ with the graph of $y = \frac{1}{2}x$.

How many solutions does $\sqrt{4x - 4} = \frac{1}{2}x$ have? two

Replace the graph of $y = \frac{1}{2}x$ with the graph of $y = 2x - 5$.

How many solutions does $\sqrt{4x - 4} = 2x - 5$ have? one

(D) Graph both sides of $\sqrt{2x - 3} = \sqrt{x}$ as separate functions on your calculator.

How many solutions does $\sqrt{2x - 3} = \sqrt{x}$ have? ______

Replace the graph of $y = \sqrt{x}$ with the graph of $y = \sqrt{2x + 3}$.

How many solutions does $\sqrt{2x - 3} = \sqrt{2x + 3}$ have? ______

(D) Graph both sides of $\sqrt{2x - 3} = \sqrt{x}$ as separate functions on your calculator.

How many solutions does $\sqrt{2x - 3} = \sqrt{x}$ have? one

Replace the graph of $y = \sqrt{x}$ with the graph of $y = \sqrt{2x + 3}$.

How many solutions does $\sqrt{2x - 3} = \sqrt{2x + 3}$ have? zero

QUESTIONING STRATEGIES

? How can the **INTERSECT** feature on your calculator help you solve a radical equation by graphing? **You can graph the function represented by each side of the equation and find the *x*-value of the point of intersection, or determine that one does not exist.**

Reflect

1. For a square root equation of the form $\sqrt{bx - h} = c$, what can you conclude about the number of solutions based on the sign of c?

2. For a square root equation of the form $\sqrt{bx - h} + k = c$, what can you conclude about the number of solutions based on the values of k and c?

Reflect

1. For a square root equation of the form $\sqrt{bx - h} = c$, what can you conclude about the number of solutions based on the sign of c?
When c is nonnegative, there is one solution. When c is negative, there are no solutions.

2. For a square root equation of the form $\sqrt{bx - h} + k = c$, what can you conclude about the number of solutions based on the values of k and c?
When $k \leq c$, there is one solution. When $k > c$, there are no solutions.

3. For a cube root equation of the form $\sqrt[3]{bx - h} = c$, will the number of solutions depend on the sign of c? Explain.

4. The graphs in the second part of Step D appear to be get closer and closer as x increases. How can you be sure that they never meet, that is, that $\sqrt{2x - 3} = \sqrt{2x + 3}$ really has no solutions?

3. For a cube root equation of the form $\sqrt[3]{bx - h} = c$, will the number of solutions depend on the sign of c? Explain.

No, a cubic expression can have negative roots.

4. The graphs in the second part of Step D appear to be get closer and closer as x increases. How can you be sure that they never meet, that is, that $\sqrt{2x - 3} = \sqrt{2x + 3}$ really has no solutions?

The graphs have the same shape. They are just different horizontal translations of $y = \sqrt{2x}$, one to the right and one to the left.

Explain 1 Solving Square Root and $\frac{1}{2}$-Power Equations

A *radical equation* contains a variable within a radical or a variable raised to a (non-integer) rational power. To solve a square root equation, or, equivalently, an equation involving the power $\frac{1}{2}$, you can square both sides of the equation and solve the resulting equation.

Because opposite numbers have the same square, squaring both sides of an equation may introduce an apparent solution that is not an actual solution (an extraneous solution). For example, while the only solution of $x = 2$ is 2, the equation that is the square of each side, $x^2 = 4$, has two solutions, -2 and 2. But -2 is not a solution of the original equation.

QUESTIONING STRATEGIES

Why do you need to square both sides of the equation? **Squaring is the inverse of taking a square root, so squaring the side that contains the radical gets rid of the radical sign. You have to square the other side, too, in order to maintain the equality.**

Why don't all of the apparent solutions check? **When you square both sides of an equation, you can introduce an extraneous, or false, solution.**

Explain 1 Solving Square Root and $\frac{1}{2}$-Power Equations

A *radical equation* contains a variable within a radical or a variable raised to a (non-integer) rational power. To solve a square root equation, or, equivalently, an equation involving the power $\frac{1}{2}$, you can square both sides of the equation and solve the resulting equation.

Because opposite numbers have the same square, squaring both sides of an equation may introduce an apparent solution that is not an actual solution (an extraneous solution). For example, while the only solution of $x = 2$ is 2, the equation that is the square of each side, $x^2 = 4$, has two solutions, -2 and 2. But -2 is not a solution of the original equation.

In this lesson, students use inverse operations to solve radical equations, which are equations in which the variable is in the radicand. For example, since the inverse of taking a cube root is raising to the third power, equations containing a cube root of a variable expression can be solved by cubing both sides of the equation. Squaring both sides of an equation may produce an extraneous solution; that is, a solution that is not a solution of the original equation.

AVOID COMMON ERRORS

When solving equations containing square roots,

AVOID COMMON ERRORS

When solving equations containing square roots, students may forget to square the entire quantity on each side. Caution students not to square terms individually when solving an equation such as $\sqrt{x+4} = x - 2$. Point out that when both sides of this equation are squared, the result is $x + 4 = (x-2)^2$, not $x + 4 = x^2 - 2^2$.

(A) $2 + \sqrt{x+10} = x$

	$2 + \sqrt{x+10} = x$
Isolate the radical.	$\sqrt{x+10} = x - 2$
Square both sides.	$\left(\sqrt{x+10}\right)^2 = (x-2)^2$
Simplify.	$x + 10 = x^2 - 4x + 4$
Simplify.	$0 = x^2 - 5x - 6$
Factor.	$0 = (x-6)(x+1)$
Zero Product Property	$x = 6$ or $x = -1$

Check:

$2 + \sqrt{x+10} = x$	$2 + \sqrt{x+10} = x$
$2 + \sqrt{6+10} \stackrel{?}{=} 6$	$2 + \sqrt{-1+10} \stackrel{?}{=} -1$
$2 + \sqrt{16} \stackrel{?}{=} 6$	$2 + \sqrt{9} \stackrel{?}{=} -1$
$6 = 6$ ✓	$5 \neq -1$
$x = 6$ is a solution.	$x = -1$ is not a solution.

The solution is $x = 6$.

(B) $(x+6)^{\frac{1}{2}} - (2x-4)^{\frac{1}{2}} = 0$

Rewrite with radicals. $\sqrt{x+6} - \sqrt{2x-4} = 0$

Isolate radicals on each side. $\sqrt{x+6} = \square$

Square both sides. $\left(\sqrt{x+6}\right)^2 = \left(\square\right)^2$

Simplify. $\square = \square$

Solve. $\square = x$

Check:

$\sqrt{x+6} - \sqrt{2x-4} = 0$

$\sqrt{10+6} - \sqrt{2(\square) - 4} \stackrel{?}{=} 0$

$\square - \square \stackrel{?}{=} 0$

$\square = 0$

(B) $(x+6)^{\frac{1}{2}} - (2x-4)^{\frac{1}{2}} = 0$

Rewrite with radicals. $\sqrt{x+6} - \sqrt{2x-4} = 0$

Isolate radicals on each side. $\sqrt{x+6} = \boxed{\sqrt{2x-4}}$

Square both sides. $\left(\sqrt{x+6}\right)^2 = \left(\boxed{\sqrt{2x-4}}\right)^2$

Simplify. $\boxed{x+6} = \boxed{2x-4}$

Solve. $\boxed{10} = x$

Check:

$\sqrt{x+6} - \sqrt{2x-4} = 0$

$\sqrt{10+6} - \sqrt{2(\boxed{10}) - 4} \stackrel{?}{=} 0$

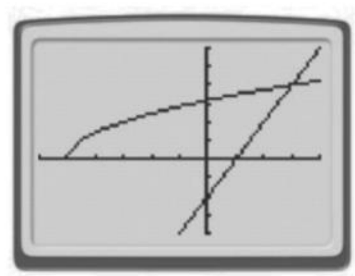
$\boxed{\sqrt{16}} - \boxed{\sqrt{16}} \stackrel{?}{=} 0$

$\boxed{0} = 0$

The solution is $x = 10$.

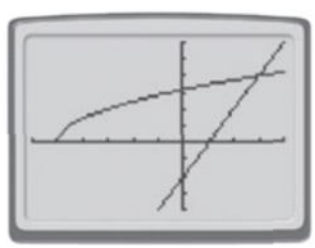
Reflect

5. The graphs of each side of the equation from Part A are shown on the graphing calculator screen below. How can you tell from the graph that one of the two solutions you found algebraically is extraneous?



Reflect

5. The graphs of each side of the equation from Part A are shown on the graphing calculator screen below. How can you tell from the graph that one of the two solutions you found algebraically is extraneous?



The graphs intersect at only 1 point, at the solution $x = 6$, so there is only one solution.

The graphs do not intersect at the apparent solution $x = -1$.

Your Turn

6. Solve $(x+5)^{\frac{1}{2}} - 2 = 1$.

Your Turn

6. Solve $(x+5)^{\frac{1}{2}} - 2 = 1$.

$$(x+5)^{\frac{1}{2}} - 2 = 1$$

$$(x+5)^{\frac{1}{2}} = 3$$

$$\left((x+5)^{\frac{1}{2}}\right)^2 = 3^2$$

$$x+5 = 9$$

$$x = 4$$

Check:

$$(4+5)^{\frac{1}{2}} - 2 \stackrel{?}{=} 1$$

$$3 - 2 \stackrel{?}{=} 1$$

$$1 = 1 \checkmark$$

The solution is $x = 4$.

Explain 2 Solving Cube Root and $\frac{1}{3}$-Power Equations

You can solve radical equations that involve roots other than square roots by raising both sides to the index of the radical. So, to solve a cube root equation, or, equivalently, an equation involving the power $\frac{1}{3}$, you can cube both sides of the equation and solve the resulting equation.

Example 2 **Solve the equation.**

(A) $\sqrt[3]{x+2}+7=5$

$$\sqrt[3]{x+2}+7=5$$

Isolate the radical. $\sqrt[3]{x+2}=-2$

Cube both sides. $\left(\sqrt[3]{x+2}\right)^3=(-2)^3$

Simplify. $x+2=-8$

Solve for x $x=-10$

The solution is $x=-10$.

QUESTIONING STRATEGIES

? If an equation contains a variable expression raised to the one-third power, can the solution process produce extraneous solutions? How do you know? **No; to solve the equation, you would need to cube both sides, and cubing both sides of an equation does not introduce the possibility of extraneous solutions.**

(B) $\sqrt[3]{x-5} = x + 1$

$$\sqrt[3]{x-5} = x + 1$$

Cube both sides. $\left(\sqrt[3]{x-5}\right)^3 = (x+1)^3$

Simplify $\square = \square$

Simplify. $0 = \square$

Begin to factor by grouping. $0 = x^2(\square) + 2(\square)$

Complete factoring $0 = (x^2 + 2)(\square)$

By the Zero Product Property, $\square = 0$ or $\square = 0$.

Because there are no real values of x for which $x^2 = \square$, the only solution is $\square$.

(B) $\sqrt[3]{x-5} = x + 1$

$$\sqrt[3]{x-5} = x + 1$$

Cube both sides. $\left(\sqrt[3]{x-5}\right)^3 = (x+1)^3$

Simplify $\boxed{x - 5} = \boxed{x^3 + 3x^2 + 3x + 1}$

Simplify. $0 = \boxed{x^3 + 3x^2 + 2x + 6}$

Begin to factor by grouping. $0 = x^2\left(\boxed{x + 3}\right) + 2\left(\boxed{x + 3}\right)$

Complete factoring $0 = (x^2 + 2)\left(\boxed{x + 3}\right)$

By the Zero Product Property, $\boxed{x^2 + 2} = 0$ or $\boxed{x + 3} = 0$.

Because there are no real values of x for which $x^2 = \boxed{-2}$, the only solution is $\boxed{x = -3}$.

Reflect

7. Discussion Example 1 shows checking for extraneous solutions, while Example 2 does not. While it is always wise to check your answers, can a cubic equation have an extraneous solution? Explain your answer.

Reflect

7. Discussion Example 1 shows checking for extraneous solutions, while Example 2 does not. While it is always wise to check your answers, can a cubic equation have an extraneous solution? Explain your answer.

No; while every positive number has two square roots, every number—positive or negative—has exactly one cube root. That is, the cube of every number is unique. So, when you cube both sides of an equation, you do not introduce the possibility of another number having the same cube as an actual solution.

Your Turn

8. Solve $2(x - 50)^{\frac{1}{3}} = -10$.

Your Turn

8. Solve $2(x - 50)^{\frac{1}{3}} = -10.$

$$2(x - 50)^{\frac{1}{3}} = -10$$

$$(x - 50)^{\frac{1}{3}} = -5$$

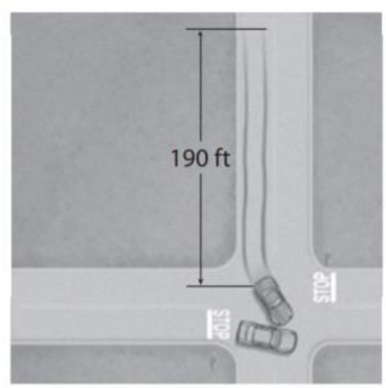
$$\left((x - 50)^{\frac{1}{3}}\right)^3 = (-5)^3$$

$$x - 50 = -125$$

$$x = -75$$

Explain 3 Solving a Real-World Problem

Ⓐ **Driving** The speed s in miles per hour that a car is traveling when it goes into a skid can be estimated by using the formula $s = \sqrt{30fd}$, where f is the coefficient of friction and d is the length of the skid marks in feet.



After an accident, a driver claims to have been traveling the speed limit of 55 mi/h. The coefficient of friction under the conditions at the time of the accident was 0.6, and the length of the skid marks is 190 feet. Is the driver telling the truth about the car's speed? Explain.

Use the formula to find the length of a skid at a speed of 55 mi/h. Compare this distance to the actual skid length of 190 feet.

$$s = \sqrt{30fd}$$

Substitute 55 for *s* and 0.6 for *f*	$55 = \sqrt{30(0.6)d}$
Simplify.	$55 = \sqrt{18d}$
Square both sides.	$55^2 = \left(\sqrt{18d}\right)^2$
Simplify.	$3025 = 18d$
Solve for *d*.	$168 \approx d$

If the driver had been traveling at 55 mi/h, the skid marks would measure about 168 feet. Because the skid marks actually measure 190 feet, the driver must have been driving faster than 55 mi/h.

QUESTIONING STRATEGIES

? What are the restrictions on the variables in a square root function that models a real-world situation? **The variables in the radicand are restricted to values that make the radicand non-negative. Also, the values of the variables are restricted to values that make sense in the given context.**

Ⓑ **Construction** The diameter d in inches of a rope needed to lift a weight of w tons is given by the formula $d = \frac{\sqrt{15w}}{\pi}$. How much weight can be lifted with a rope with a diameter of 1.0 inch?

Use the formula for the diameter as a function of weight, and solve for the weight given the diameter.

$$d = \frac{\sqrt{15w}}{\pi}$$

Substitute. $\square = \frac{\sqrt{15w}}{\pi}$

Square both sides. $\square = \square$

Isolate the radical. $\left(\square\right)^2 = \left(\sqrt{15w}\right)^2$

Simplify. $\square = 15w$

Solve for w. $\square \approx w$

A rope with a diameter of 1.0 can hold about ______ ton, or about ______ pounds.

Ⓑ **Construction** The diameter d in inches of a rope needed to lift a weight of w tons is given by the formula $d = \frac{\sqrt{15w}}{\pi}$. How much weight can be lifted with a rope with a diameter of 1.0 inch?

Use the formula for the diameter as a function of weight, and solve for the weight given the diameter.

$$d = \frac{\sqrt{15w}}{\pi}$$

Substitute. $\boxed{1.0} = \frac{\sqrt{15w}}{\pi}$

Square both sides. $\boxed{\pi} = \boxed{\sqrt{15w}}$

Isolate the radical. $\left(\boxed{\pi}\right)^2 = \left(\sqrt{15w}\right)^2$

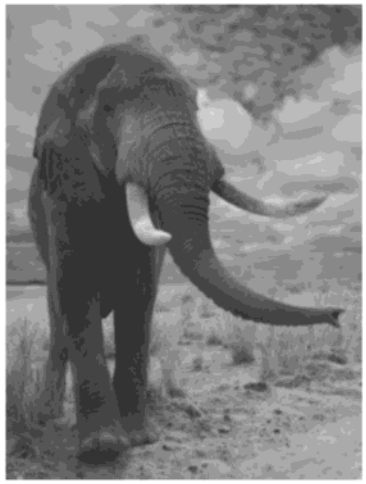
Simplify. $\boxed{\pi^2} = 15w$

Solve for w. $\boxed{0.66} \approx w$

A rope with a diameter of 1.0 can hold about **0.66** ton, or about **1320** pounds.

Your Turn

9. **Biology** The trunk length (in inches) of a male elephant can be modeled by $l = 23\sqrt[3]{t} + 17$, where t is the age of the elephant in years. If a male elephant has a trunk length of 100 inches, about what is his age?



Your Turn

9. **Biology** The trunk length (in inches) of a male elephant can be modeled by $l = 23\sqrt[3]{t} + 17$, where t is age of the elephant in years. If a male elephant has a trunk length of 100 inches, about what is his age?

$$l = 23\sqrt[3]{t} + 17$$
$$100 = 23\sqrt[3]{t} + 17$$
$$83 = 23\sqrt[3]{t}$$
$$3.61 = \sqrt[3]{t}$$
$$(3.61)^3 = \left(\sqrt[3]{t}\right)^3$$
$$47 \approx t$$

The elephant is about 47 years old.

10. A student asked to solve the equation $\sqrt{4x+8}+9=1$ isolated the radical, squared both sides, and solved for x to obtain $x=14$, only to find out that the apparent solution was extraneous. Why could the student have stopped trying to solve the equation after isolating the radical?

10. A student asked to solve the equation $\sqrt{4x+8}+9=1$ isolated the radical, squared both sides, and solved for x to obtain $x=14$, only to find out that the apparent solution was extraneous. Why could the student have stopped trying to solve the equation after isolating the radical?

Isolating the radical gives $\sqrt{4x+8}=-8$. Because the principal square root of a quantity cannot be negative, it is clear that there will be no real solution.

11. When you see a cube root equation with the radical expression isolated on one side and a constant on the other, what should you expect for the number of solutions? Explain. What are some reasons you should check your answer anyway?

11. When you see a cube root equation with the radical expression isolated on one side and a constant on the other, what should you expect for the number of solutions? Explain. What are some reasons you should check your answer anyway?

There will always be one solution since there is a unique cube root for every real number. You should check your answer anyway, even though there won't be extraneous solutions, to make sure you haven't made any computational mistakes. Also, in a real-world context, you need to make sure the answer makes sense in the situation.

12. **Essential Question Check-In** Solving a quadratic equation of the form $x^2 = a$ involves taking the square root of both sides. Solving a square root equation of the form $\sqrt{x} = b$ involves squaring both sides of the equation. Which of these operations can create an equation that is not equivalent to the original equation? Explain how this affects the solution process.

12. **Essential Question Check-In** Solving a quadratic equation of the form $x^2 = a$ involves taking the square root of both sides. Solving a square root equation of the form $\sqrt{x} = b$ involves squaring both sides of the equation. Which of these operations can create an equation that is not equivalent to the original equation? Explain how this affects the solution process.

Squaring both sides; squaring both sides can create an apparent solution that is not a solution of the original equation, that is, an extraneous solution. This means that you must be sure to check for extraneous solutions.

SUMMARIZE THE LESSON

How do you solve an equation that contains a radical? **First, isolate the radical. Then raise both sides of the equation to the power equal to the index of the radical. Finally, solve the resulting equation for the variable, and check for extraneous solutions.**