Introduction to Calculus

11

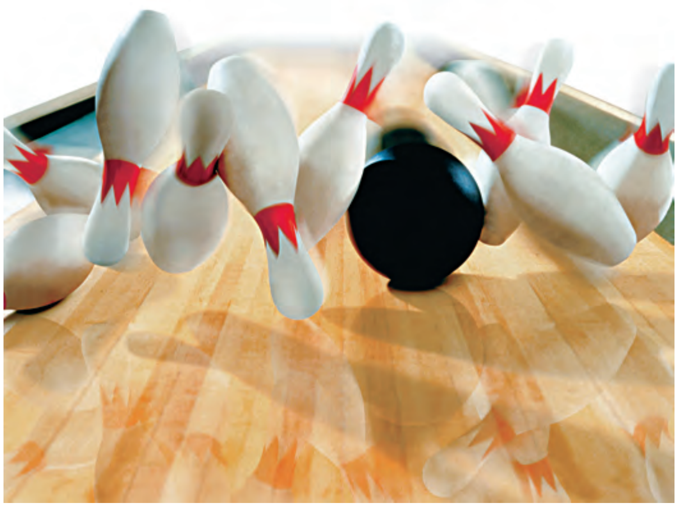
We revisit an idea introduced in an essay in Chapter 1. Take a rapid sequence of still photographs of a moving scene and project them onto a screen at thirty shots a second or faster. Our eyes see the result as continuous motion. The small difference between one frame and the next cannot be detected by the human visual system. The idea of calculus likewise regards continuous motion as made up of a sequence of still configurations. In this chapter, you will see how calculus uses a revolutionary concept called *limits* to master the mystery of movement by "freezing the frame" instant by instant.

Using limits to describe instantaneous rates of change is introduced in the Section 11.4 opener and developed throughout the section.

Section 11.1 Finding Limits Using Tables and Graphs

Objectives

1. Understand limit notation.
2. Find limits using tables.
3. Find limits using graphs.
4. Find one-sided limits and use them to determine if a limit exists.



Motion and change are the very essence of life. Moving air brushes against our faces, rain falls on our heads, birds fly past us, plants spring from the earth, grow, and then die, and rocks thrown upward reach a maximum height before falling to the ground.

The tools of algebra and trigonometry are essentially static; numbers, points, lines, equations, functions, and graphs do not incorporate motion. The development of calculus in the middle of the seventeenth century provided a way to use these static tools to analyze motion and change. It took nearly two thousand years of effort for humankind to achieve this feat, made possible by a revolutionary concept called *limits*. The invention of limits marked a turning point in human history, having as dramatic an impact on our lives as the invention of the wheel and the printing press. In this section, we introduce this bold and dramatic style of thinking about mathematics.

1 Understand limit notation.

An Introduction to Limits

Suppose that you and a friend are walking along the graph of the function

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

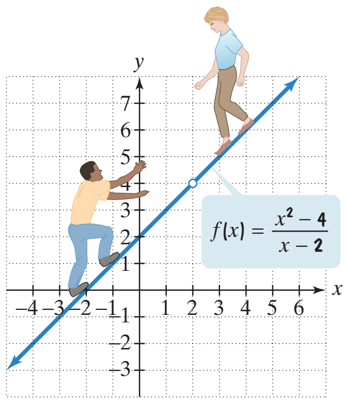


Figure 11.1 Walking along the graph of f, very close to 2

Figure 11.1 illustrates that you are walking uphill and your friend is walking downhill. Because 2 is not in the domain of the function, there is a hole in the graph at $x = 2$. Warning signs along the graph might be appropriate: Caution: $f(2)$ is undefined! If you or your friend reach 2, you will fall through the hole and splatter onto the x-axis.

Obviously, there is a problem at $x = 2$. But what happens along the graph of $f(x) = \frac{x^2 - 4}{x - 2}$ as you and your friend walk very, very close to $x = 2$? What function value, $f(x)$, will each of you approach? One way to answer this question is to construct a table of function values to analyze numerically the behavior of f as x gets closer and closer to 2. Remember that you are walking uphill, approaching 2 from the left side of 2. Your friend is walking downhill, approaching 2 from the right side of 2. Thus, we must include values of x that are less than 2 and values of x that are greater than 2.

In **Table 11.1** at the top of the next page, we choose values of x close to 2. As x approaches 2 from the left, we arbitrarily start with $x = 1.99$. Then we select two additional values of x that are closer to 2, but still less than 2. We choose 1.999 and 1.9999. As x approaches 2 from the right, we arbitrarily start with $x = 2.01$. Then we select two additional values of x that are closer to 2, but still greater than 2. We choose 2.001 and 2.0001. Finally, evaluate f at each chosen value of x to obtain **Table 11.1**.

Technology

A graphing utility with a TABLE feature can be used to generate the entries in Table 11.1. In TBLSET, change Auto to Ask for Indpnt, the independent variable. Here is a typical screen that verifies **Table 11.1**.

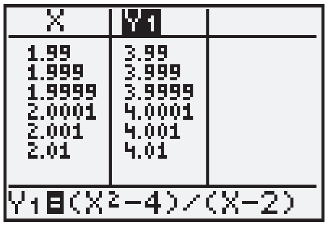


Table 11.1

	x approaches 2 from the left.				x approaches 2 from the right.		
x	1.99	1.999	1.9999	2	2.0001	2.001	2.01
$f(x) = \dfrac{x^2 - 4}{x - 2}$	3.99	3.999	3.9999	Undefined	4.0001	4.001	4.01
	$f(x)$ gets closer to 4.				$f(x)$ gets closer to 4.		

From **Table 11.1**, it appears that as x gets closer to 2, the values of $f(x) = \dfrac{x^2 - 4}{x - 2}$ get closer to 4. We say that

$$\text{“The limit of } \frac{x^2 - 4}{x - 2} \text{ as } x \text{ approaches 2 equals the number 4.”}$$

We can express this sentence in a mathematical notation called **limit notation**. We use an arrow for the word *approaches*. Likewise, we use *lim* as shorthand for the word *limit*. Thus, the limit notation for the English sentence in quotations is

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4. \qquad \text{The limit of } \frac{x^2 - 4}{x - 2} \text{ as } x \text{ approaches 2 equals the number 4.}$$

Calculus is the study of limits and their applications. Limits are the foundation of the concepts that you will encounter in calculus.

Limit Notation and Its Description

Suppose that f is a function defined on some open interval containing the number a. The function f may or may not be defined at a. The **limit notation**

$$\lim_{x \to a} f(x) = L$$

is read “the limit of $f(x)$ as x approaches a equals the number L.” This means that as x gets closer to a, but remains unequal to a, the corresponding values of $f(x)$ get closer to L.

2 Find limits using tables.

Finding Limits Using Tables

To find $\lim\limits_{x \to a} f(x)$, use a graphing utility with a TABLE feature or create a table by hand. Approach a from the left, choosing values of x that are close to a, but still less than a. Then approach a from the right, choosing values of x that are close to a, but still greater than a. Evaluate f at each chosen value of x to obtain the desired table.

Choose values of x so that the table makes it obvious what the corresponding values of $f(x)$ are getting close to. If the values of $f(x)$ are getting close to the number L, we infer that

$$\lim_{x \to a} f(x) = L.$$

EXAMPLE 1 Finding a Limit Using a Table

Find: $\lim_{x \to 4} 3x^2$.

Solution As x gets closer to 4, but remains unequal to 4, we must find the number that the corresponding values of $3x^2$ get closer to. The voice balloons shown below indicate that in this limit problem, $f(x) = 3x^2$ and $a = 4$.

$$\lim_{x \to 4} 3x^2$$

This is a in $\lim_{x \to a} f(x)$.

This is $f(x)$ in $\lim_{x \to a} f(x)$.

In making a table, we choose values of x close to 4. As x approaches 4 from the left, we arbitrarily start with $x = 3.99$. Then we select two additional values of x that are closer to 4, but still less than 4. We choose 3.999 and 3.9999. As x approaches 4 from the right, we arbitrarily start with $x = 4.01$. Then we select two additional numbers that are closer to 4, but still greater than 4. We choose 4.001 and 4.0001. Finally, we evaluate f at each chosen value of x to obtain **Table 11.2**. The values of $f(x)$ in the table are rounded to four decimal places.

Table 11.2

	x approaches 4 from the left.				x approaches 4 from the right.		
x	3.99	3.999	3.9999	$\longrightarrow\ \longleftarrow$	4.0001	4.001	4.01
$f(x) = 3x^2$	47.7603	47.9760	47.9976	$\longrightarrow\ \longleftarrow$	48.0024	48.0240	48.2403
	$f(x)$ gets closer to 48.				$f(x)$ gets closer to 48.		

From **Table 11.2**, it appears that as x gets closer to 4, the values of $3x^2$ get closer to 48. We infer that

$$\lim_{x \to 4} 3x^2 = 48.$$

Check Point 1 Find: $\lim_{x \to 3} 4x^2$.

EXAMPLE 2 Finding a Limit Using a Table

Find: $\lim_{x \to 0} \frac{\sin x}{x}$.

Solution As x gets closer to 0, but remains unequal to 0, we must find the number that the corresponding values of $\frac{\sin x}{x}$ get closer to. The voice balloons shown below indicate that in this limit problem, $f(x) = \frac{\sin x}{x}$ and $a = 0$.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

This is a in $\lim_{x \to a} f(x)$.

This is $f(x)$ in $\lim_{x \to a} f(x)$.

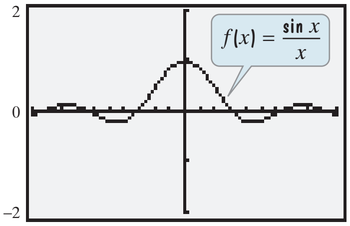
Because division by 0 is undefined, the domain of $f(x) = \frac{\sin x}{x}$ is $\{x \mid x \neq 0\}$. Thus, f is not defined at 0. However, in this limit problem, we do not care what is happening at $x = 0$. We are interested in the behavior of the function as x gets close to 0. **Table 11.3** shows the values of $f(x)$, rounded to five decimal places, as x approaches 0 from the left and from the right. Values of x in the table are measured in radians.

Technology

Graphic Connections

The graph of $f(x) = \frac{\sin x}{x}$ illustrates that as x gets closer to 0, the values of $f(x)$ are approaching 1. This supports our inference that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$



[−10, 10, 1] by [−2, 2, 1]

Table 11.3

	x approaches 0 from the left.				x approaches 0 from the right.		
x	−0.03	−0.02	−0.01	→ ←	0.01	0.02	0.03
$f(x) = \frac{\sin x}{x}$	0.99985	0.99993	0.99998	→ ←	0.99998	0.99993	0.99985
	$f(x)$ gets closer to 1.				$f(x)$ gets closer to 1.		

From **Table 11.3**, it appears that as x gets closer to 0, the values of $\frac{\sin x}{x}$ get closer to 1. We infer that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Check Point 2 Find: $\lim_{x \to 0} \frac{\cos x - 1}{x}$.

3 Find limits using graphs.

Finding Limits Using Graphs

The limit statement

$$\lim_{x \to a} f(x) = L$$

is illustrated in **Figure 11.2**. In the three graphs, the number that x is approaching, a, is shown on the x-axis. The limit, L, is shown on the y-axis. Take a few minutes to examine the graphs. Can you see that as x approaches a along the x-axis, $f(x)$ approaches L along the y-axis? In each graph, as x gets closer to a, the values of $f(x)$ get closer to L.

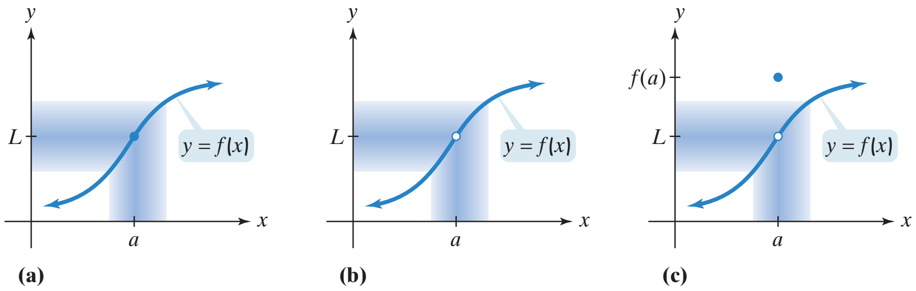


Figure 11.2 In each graph, as x gets closer to a, the values of f get closer to L: $\lim_{x \to a} f(x) = L$.

In **Figure 11.2(a)**, as x approaches a, $f(x)$ approaches L. At a, the value of the function is L: $f(a) = L$. In **Figure 11.2(b)**, as x approaches a, $f(x)$ approaches L. This is true although f is not defined at a, shown by the hole in the graph. In **Figure 11.2(c)**, we again see that as x approaches a, $f(x)$ approaches L. Notice, however, that the value of the function at a, $f(a)$, shown by the blue dot, is not equal to the limit: $f(a) \neq L$. What you get as you approach a is not the same as what you get at a.

Example 3 illustrates that the graph of a function can sometimes be helpful in finding limits.

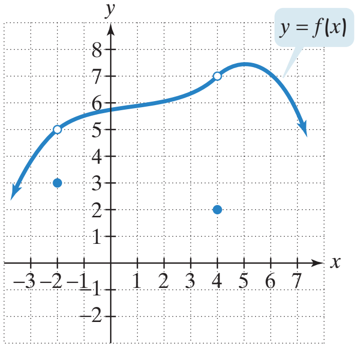


Figure 11.3

EXAMPLE 3 Finding a Limit Using a Graph

Use the graph in **Figure 11.3** to find each of the following:

a. $\lim_{x \to 4} f(x)$ **b.** $f(4)$.

Solution

a. To find $\lim_{x \to 4} f(x)$, examine the graph of f *near* $x = 4$. As x gets closer to 4, the values of $f(x)$ get closer to the y-coordinate of the point shown by the open dot on the right. The y-coordinate of this point is 7. Thus, as x gets closer to 4, the values of $f(x)$ get closer to 7. We conclude from the graph that

$$\lim_{x \to 4} f(x) = 7.$$

b. To find $f(4)$, examine the graph of f *at* $x = 4$. At $x = 4$, the open dot is not included in the graph of f. The graph of f at 4 is shown by the closed dot with coordinates $(4, 2)$. Thus, $f(4) = 2$.

In Example 3, notice that the value of f at 4 has nothing to do with the conclusion that $\lim_{x \to 4} f(x) = 7$. Regardless of how f is defined at 4, it is still true that $\lim_{x \to 4} f(x) = 7$. Furthermore, if f were undefined at 4, the limit of $f(x)$ as $x \to 4$ would still equal 7.

Check Point 3 Use the graph in **Figure 11.3** to find each of the following:

a. $\lim_{x \to -2} f(x)$ **b.** $f(-2)$.

EXAMPLE 4 Finding a Limit by Graphing a Function

Graph the function

$$f(x) = \begin{cases} 2x - 4 & \text{if } x \neq 3 \\ -5 & \text{if } x = 3. \end{cases}$$

Use the graph to find $\lim_{x \to 3} f(x)$.

Solution This piecewise function is defined by two equations. Graph the piece defined by the linear function, $f(x) = 2x - 4$, using the y-intercept, -4, and the slope, 2. Because $x = 3$ is not included, show an open dot on the line corresponding to $x = 3$. This open dot, with coordinates $(3, 2)$, is shown in **Figure 11.4**.

Now we complete the graph using $f(x) = -5$ if $x = 3$. This part of the function is graphed as the point $(3, -5)$, shown as a closed blue dot in **Figure 11.4**.

To find $\lim_{x \to 3} f(x)$, examine the graph of f near $x = 3$. As x gets closer to 3, the values of $f(x)$ get closer to the y-coordinate of the point shown by the open dot. The y-coordinate of this point is 2. We conclude from the graph that

$$\lim_{x \to 3} f(x) = 2.$$

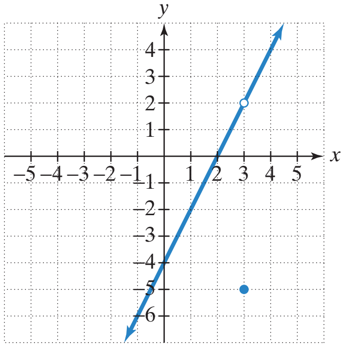


Figure 11.4 As x gets closer to 3, what number are the function values getting closer to?

Check Point 4 Graph the function

$$f(x) = \begin{cases} 3x - 2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$$

Use the graph to find $\lim_{x \to 2} f(x)$.

4 Find one-sided limits and use them to determine if a limit exists.

One-Sided Limits

The graph in **Figure 11.5** shows a portion of the graph of the function $f(x) = x^2$. The graph illustrates that

$$\lim_{x \to 2} x^2 = 4.$$

As x gets closer to 2, but remains unequal to 2, the corresponding values of $f(x)$ get closer to 4. The values of x near 2 fall into two categories: those that lie to the left of 2, shown by the red arrow on the x-axis, and those that lie to the right of 2, shown by the blue arrow on the x-axis.

The values of x can get closer to 2 in two ways. The values of x can approach 2 from the left, through numbers that are less than 2. **Table 11.4** shows some values of x and the corresponding values of $f(x)$ rounded to four decimal places. The red portion of the graph in **Figure 11.5** shows that as x approaches 2 from the left of 2, $f(x)$ approaches 4.

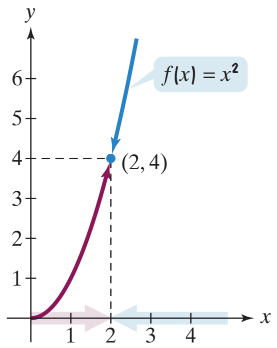


Figure 11.5 As x approaches 2 from the left (red arrow) or from the right (blue arrow), values of $f(x)$ get closer to 4.

Table 11.4

x	1.99	1.999	1.9999 →
$f(x) = x^2$	3.9601	3.9960	3.9996 →

We say that "the limit of x^2 as x approaches 2 from the left equals 4." The mathematical notation for this English sentence is

$$\lim_{x \to 2^-} x^2 = 4.$$

The notation $x \to 2^-$ indicates that x is less than 2 and is approaching 2 from the left.

The values of x can also approach 2 from the right, through numbers that are greater than 2. **Table 11.5** shows some values of x and the corresponding values of $f(x)$ rounded to four decimal places. The blue portion of the graph in **Figure 11.5** shows that as x approaches 2 from the right of 2, $f(x)$ approaches 4.

Table 11.5

x	← 2.0001	2.001	2.01
$f(x) = x^2$	← 4.0004	4.0040	4.0401

We say that "the limit of x^2 as x approaches 2 from the right equals 4." The mathematical notation for this English sentence is

$$\lim_{x \to 2^+} x^2 = 4.$$

The notation $x \to 2^+$ indicates that x is greater than 2 and is approaching 2 from the right.

In general, if x approaches a from one side, we have a **one-sided limit**.

One-Sided Limits

Left-Hand Limit The limit notation

$$\lim_{x \to a^-} f(x) = L$$

is read "the limit of $f(x)$ as x approaches a from the left equals L" and is called the **left-hand limit**. This means that as x gets closer to a, but remains less than a, the corresponding values of $f(x)$ get closer to L.

Right-Hand Limit The limit notation

$$\lim_{x \to a^+} f(x) = L$$

is read "the limit of $f(x)$ as x approaches a from the right equals L" and is called the **right-hand limit**. This means that as x gets closer to a, but remains greater than a, the corresponding values of $f(x)$ get closer to L.

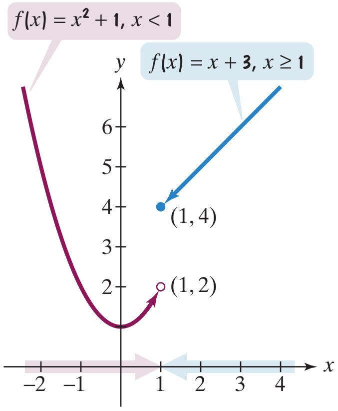


Figure 11.6 As x approaches 1 from the left (red arrow) and from the right (blue arrow), values of $f(x)$ do not get closer to a single number.

A function's graph can be helpful in finding one-sided limits. For example, **Figure 11.6** shows the graph of the piecewise function

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 3 & \text{if } x \geq 1. \end{cases}$$

The red portion of the graph, part of a parabola, illustrates that as x approaches 1 from the left, the corresponding values of $f(x)$ get closer to 2. The left-hand limit is 2:

$$\lim_{x \to 1^-} f(x) = 2.$$

The blue portion of the graph, part of a line, illustrates that as x approaches 1 from the right, the corresponding values of $f(x)$ get closer to 4. The right-hand limit is 4:

$$\lim_{x \to 1^+} f(x) = 4.$$

Because $\lim_{x \to 1^-} f(x) = 2$ and $\lim_{x \to 1^+} f(x) = 4$, there is no single number that the values of $f(x)$ are close to when x is close to 1. In this case, we say that ***f* has no limit as *x* approaches 1** or that $\boldsymbol{\lim_{x \to 1} f(x)}$ **does not exist**.

In general, a function f has a limit as x approaches a if and only if the left-hand limit equals the right-hand limit.

Equal and Unequal One-Sided Limits

- One-sided limits can be used to show that a function has a limit as x approaches a.

$$\lim_{x \to a} f(x) = L \text{ if and only if both}$$
$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L.$$

- One-sided limits can be used to show that a function has no limit as x approaches a.

$$\text{If } \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = M, \text{ where } L \neq M,$$
$$\lim_{x \to a} f(x) \text{ does not exist.}$$

Study Tip

The word *from* is helpful in distinguishing left- and right-hand limits. A left-hand limit means you approach the given x-value *from* the left. It does not mean that you approach toward the left on the graph. A right-hand limit means you approach the given x-value *from* the right. It does not mean you approach toward the right on the graph.

EXAMPLE 5 Finding One-Sided Limits Using a Graph

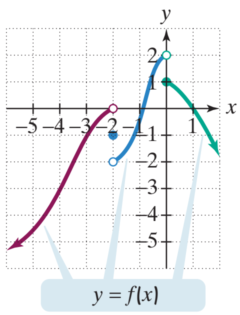


Figure 11.7

Use the graph of the piecewise function f in **Figure 11.7** to find each of the following, or state that the limit or function value does not exist:

a. $\lim_{x \to -2^-} f(x)$ **b.** $\lim_{x \to -2^+} f(x)$ **c.** $\lim_{x \to -2} f(x)$ **d.** $f(-2)$.

Solution

a. To find $\lim_{x \to -2^-} f(x)$, examine the portion of the graph shown in red that is near, but to the left of $x = -2$. As x approaches -2 from the left, the values of $f(x)$ get close to the y-coordinate of the point shown by the red open dot. This point, $(-2, 0)$, has a y-coordinate of 0. Thus,

$$\lim_{x \to -2^-} f(x) = 0.$$

b. To find $\lim_{x \to -2^+} f(x)$, examine the portion of the graph shown in blue that is near, but to the right of $x = -2$. As x approaches -2 from the right, the values of

$f(x)$ get close to the y-coordinate of the point shown by the blue open dot. This point, $(-2, -2)$, has a y-coordinate of -2. Thus,

$$\lim_{x \to -2^+} f(x) = -2.$$

c. We found that

$$\lim_{x \to -2^-} f(x) = 0 \quad \text{and} \quad \lim_{x \to -2^+} f(x) = -2.$$

The limit as x approaches -2 from the left equals 0.

The limit as x approaches -2 from the right equals -2.

Because the left- and right-hand limits are unequal, $\lim_{x \to -2} f(x)$ does not exist.

d. To find $f(-2)$, examine the graph of f *at* $x = -2$. The graph of f at -2 is shown by the blue closed dot with coordinates $(-2, -1)$. Thus, $f(-2) = -1$.

Check Point 5 Use the graph of the piecewise function f in **Figure 11.7** to find each of the following, or state that the limit or function value does not exist:

a. $\lim_{x \to 0^-} f(x)$ **b.** $\lim_{x \to 0^+} f(x)$ **c.** $\lim_{x \to 0} f(x)$ **d.** $f(0)$.

Exercise Set 11.1

Practice Exercises

In Exercises 1–4, use each table to find the indicated limit.

1. $\lim_{x \to 2} 2x^2$

x	1.99	1.999	1.9999 →	← 2.0001	2.001	2.01
$f(x) = 2x^2$	7.9202	7.9920	7.9992 →	← 8.0008	8.0080	8.0802

2. $\lim_{x \to 3} 5x^2$

x	2.99	2.999	2.9999 →	← 3.0001	3.001	3.01
$f(x) = 5x^2$	44.701	44.970	44.997 →	← 45.003	45.030	45.301

3. $\lim_{x \to 0} \dfrac{\sin 3x}{x}$

x	−0.03	−0.02	−0.01 →	← 0.01	0.02	0.03
$f(x) = \dfrac{\sin 3x}{x}$	2.9960	2.9982	2.9996 →	← 2.9996	2.9982	2.9960

4. $\lim_{x \to 0} \dfrac{\sin 4x}{\sin 2x}$

x	−0.03	−0.02	−0.01 →	← 0.01	0.02	0.03
$f(x) = \dfrac{\sin 4x}{\sin 2x}$	1.9964	1.9984	1.9996 →	← 1.9996	1.9984	1.9964

In Exercises 5–18, construct a table to find the indicated limit.

5. $\lim_{x \to 2} 5x^2$

6. $\lim_{x \to 2} (x^2 - 1)$

7. $\lim_{x \to 3} \dfrac{1}{x - 2}$

8. $\lim_{x \to 4} \dfrac{1}{x - 3}$

9. $\lim_{x \to 0} \dfrac{x}{x^2 + 1}$

10. $\lim_{x \to 0} \dfrac{x + 1}{x^2 + 1}$

11. $\lim_{x \to -2} \dfrac{x^3 + 8}{x + 2}$

12. $\lim_{x \to -5} \dfrac{x^2 - 25}{x + 5}$

13. $\lim_{x \to 0} \dfrac{2x^2 + x}{\sin x}$

14. $\lim_{x \to 0} \dfrac{\sin x^2}{x}$

15. $\lim_{x \to 0} \dfrac{\tan x}{x}$

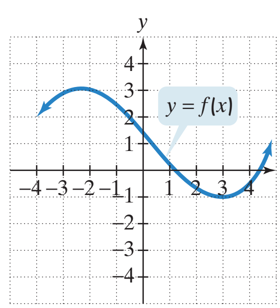
16. $\lim_{x \to 0} \dfrac{x^2}{\sec x - 1}$

17. $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$

18. $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 3x + 2 & \text{if } x \geq 0 \end{cases}$

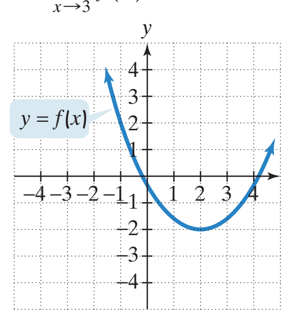
In Exercises 19–22, use the graph of f to find the indicated limit and function value.

19.



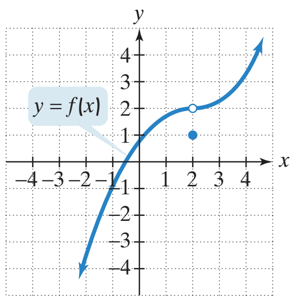
a. $\lim_{x \to 3} f(x)$ **b.** $f(3)$

20.



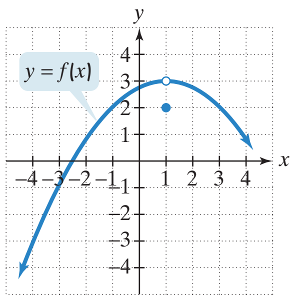
a. $\lim_{x \to 2} f(x)$ **b.** $f(2)$

21.



a. $\lim_{x\to 2} f(x)$ **b.** $f(2)$

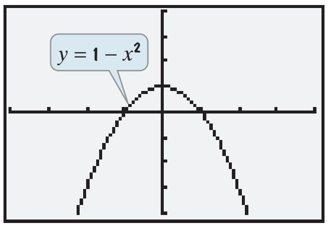
22.



a. $\lim_{x\to 1} f(x)$ **b.** $f(1)$

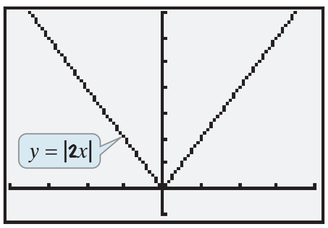
In Exercises 23–26, use the graph and the viewing rectangle shown below the graph to find the indicated limit.

23. $\lim_{x\to 2}(1 - x^2)$



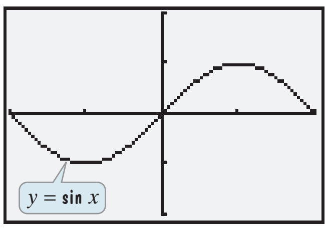
$[-4, 4, 1]$ by $[-4, 4, 1]$

24. $\lim_{x\to -2}|2x|$



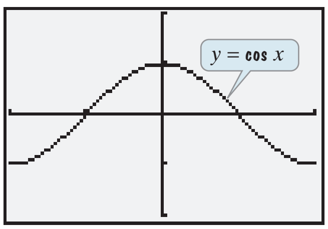
$[-4, 4, 1]$ by $[-1, 7, 1]$

25. $\lim_{x\to -\frac{\pi}{2}} \sin x$



$\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$

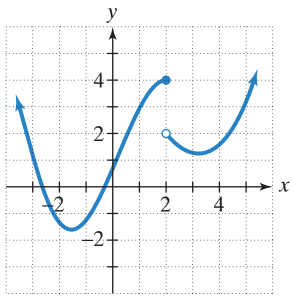
26. $\lim_{x\to -\frac{\pi}{2}} \cos x$



$\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$

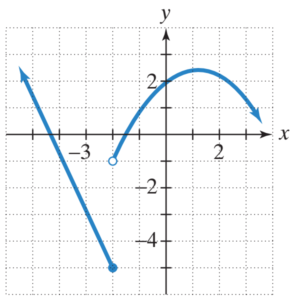
In Exercises 27–32, the graph of a function is given. Use the graph to find the indicated limits and function values, or state that the limit or function value does not exist.

27.



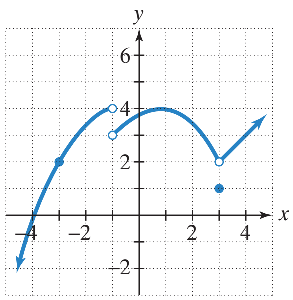
a. $\lim_{x\to 2^-} f(x)$ **b.** $\lim_{x\to 2^+} f(x)$ **c.** $\lim_{x\to 2} f(x)$ **d.** $f(2)$

28.



a. $\lim_{x\to -2^-} f(x)$ **b.** $\lim_{x\to -2^+} f(x)$ **c.** $\lim_{x\to -2} f(x)$ **d.** $f(-2)$

29.

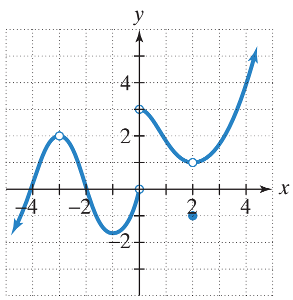


a. $\lim_{x\to -3^-} f(x)$ **b.** $\lim_{x\to -3^+} f(x)$ **c.** $\lim_{x\to -3} f(x)$ **d.** $f(-3)$

e. $\lim_{x\to -1^-} f(x)$ **f.** $\lim_{x\to -1^+} f(x)$ **g.** $\lim_{x\to -1} f(x)$ **h.** $f(-1)$

i. $\lim_{x\to 3^-} f(x)$ **j.** $\lim_{x\to 3^+} f(x)$ **k.** $\lim_{x\to 3} f(x)$ **l.** $f(3)$

30.

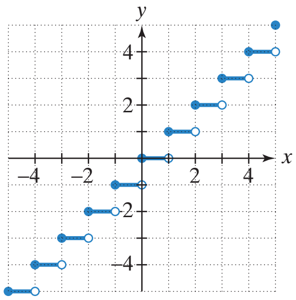


a. $\lim_{x\to -3^-} f(x)$ **b.** $\lim_{x\to -3^+} f(x)$ **c.** $\lim_{x\to -3} f(x)$ **d.** $f(-3)$

e. $\lim_{x\to 0^-} f(x)$ **f.** $\lim_{x\to 0^+} f(x)$ **g.** $\lim_{x\to 0} f(x)$ **h.** $f(0)$

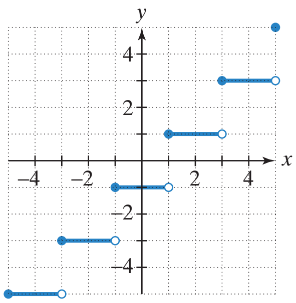
i. $\lim_{x\to 2^-} f(x)$ **j.** $\lim_{x\to 2^+} f(x)$ **k.** $\lim_{x\to 2} f(x)$ **l.** $f(2)$

31.



a. $\lim_{x\to 2^-} f(x)$ **b.** $\lim_{x\to 2^+} f(x)$ **c.** $\lim_{x\to 2} f(x)$ **d.** $f(2)$
e. $\lim_{x\to 2.5^-} f(x)$ **f.** $\lim_{x\to 2.5^+} f(x)$ **g.** $\lim_{x\to 2.5} f(x)$ **h.** $f(2.5)$

32.



a. $\lim_{x\to 3^-} f(x)$ **b.** $\lim_{x\to 3^+} f(x)$ **c.** $\lim_{x\to 3} f(x)$ **d.** $f(3)$
e. $\lim_{x\to 3.5^-} f(x)$ **f.** $\lim_{x\to 3.5^+} f(x)$ **g.** $\lim_{x\to 3.5} f(x)$ **h.** $f(3.5)$

In Exercises 33–54, graph each function. Then use your graph to find the indicated limit, or state that the limit does not exist.

33. $f(x) = 2x + 1, \lim_{x\to 3} f(x)$ **34.** $f(x) = 2x - 1, \lim_{x\to 3} f(x)$

35. $f(x) = 4 - x^2, \lim_{x\to -3} f(x)$ **36.** $f(x) = 9 - x^2, \lim_{x\to -2} f(x)$

37. $f(x) = |x + 1|, \lim_{x\to -1} f(x)$ **38.** $f(x) = |x + 2|, \lim_{x\to -2} f(x)$

39. $f(x) = \dfrac{1}{x}, \lim_{x\to -1} f(x)$ **40.** $f(x) = \dfrac{1}{x^2}, \lim_{x\to -1} f(x)$

41. $f(x) = \dfrac{x^2 - 1}{x - 1}, \lim_{x\to 1} f(x)$ **42.** $f(x) = \dfrac{x^2 - 100}{x - 10}, \lim_{x\to 10} f(x)$

43. $f(x) = e^x, \lim_{x\to 0} f(x)$ **44.** $f(x) = \ln x, \lim_{x\to 1} f(x)$

45. $f(x) = \sin x, \lim_{x\to \pi} f(x)$ **46.** $f(x) = \cos x, \lim_{x\to \pi} f(x)$

47. $f(x) = \begin{cases} x + 1 & \text{if } x \neq 2 \\ 5 & \text{if } x = 2, \end{cases} \lim_{x\to 2} f(x)$

48. $f(x) = \begin{cases} x - 1 & \text{if } x \neq 3 \\ 4 & \text{if } x = 3, \end{cases} \lim_{x\to 3} f(x)$

49. $f(x) = \begin{cases} x + 3 & \text{if } x < 0 \\ 4 & \text{if } x \geq 0, \end{cases} \lim_{x\to 0} f(x)$

50. $f(x) = \begin{cases} x + 4 & \text{if } x < 0 \\ 5 & \text{if } x \geq 0, \end{cases} \lim_{x\to 0} f(x)$

51. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1, \end{cases} \lim_{x\to 1} f(x)$

52. $f(x) = \begin{cases} 3x & \text{if } x < 1 \\ x + 2 & \text{if } x \geq 1, \end{cases} \lim_{x\to 1} f(x)$

53. $f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ \sin x & \text{if } x \geq 0, \end{cases} \lim_{x\to 0} f(x)$

54. $f(x) = \begin{cases} x & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0, \end{cases} \lim_{x\to 0} f(x)$

Practice Plus

In Exercises 55–56, use the equations for the functions f and g to graph the function $y = (f \circ g)(x)$. Then use the graph of $f \circ g$ to find the indicated limit.

55. $f(x) = x^2 - 5, g(x) = \sqrt{x}; \lim_{x\to 2}(f \circ g)(x)$

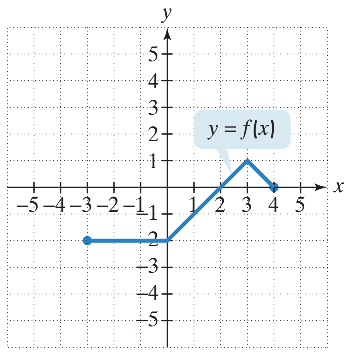
56. $f(x) = x^2 + 3, g(x) = \sqrt{x}; \lim_{x\to 1}(f \circ g)(x)$

In Exercises 57–58, use the equation for the function f to find and graph the function $y = f^{-1}(x)$. Then use the graph of f^{-1} to find the indicated limit.

57. $f(x) = x^3 - 2; \lim_{x\to 6} f^{-1}(x)$

58. $f(x) = x^3 - 4; \lim_{x\to 4} f^{-1}(x)$

In Exercises 59–66, use the graph of $y = f(x)$ to graph each function g. Then use the graph of g to find the indicated limit.



59. $g(x) = f(x) + 2; \lim_{x\to 3} g(x)$ **60.** $g(x) = f(x) - 2; \lim_{x\to 3} g(x)$

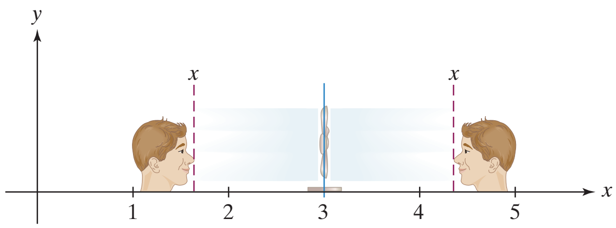
61. $g(x) = f(x + 3); \lim_{x\to 1^-} g(x)$ **62.** $g(x) = f(x + 2); \lim_{x\to 2^-} g(x)$

63. $g(x) = -f(x); \lim_{x\to -3^+} g(x)$ **64.** $g(x) = -2f(x); \lim_{x\to -3^+} g(x)$

65. $g(x) = f(2x); \lim_{x\to 1} g(x)$ **66.** $g(x) = f\left(\frac{1}{2}x\right); \lim_{x\to 1} g(x)$

Application Exercises

67. You are approaching a fan located at 3 on the x-axis.



The function f describes the breeze that you feel, $f(x)$, in miles per hour, when your nose is at position x on the x-axis. Use the values in the table to solve this exercise.

x	2.9	2.99	2.999 → ← 3.001	3.01	3.1
$f(x)$	7.7	7.92	7.991 → ← 7.991	7.92	7.7

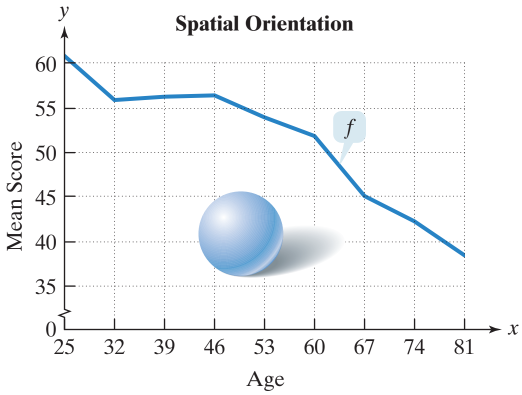
a. Find $\lim_{x\to 3} f(x)$. Describe what this means in terms of the location of your nose and the breeze that you feel.

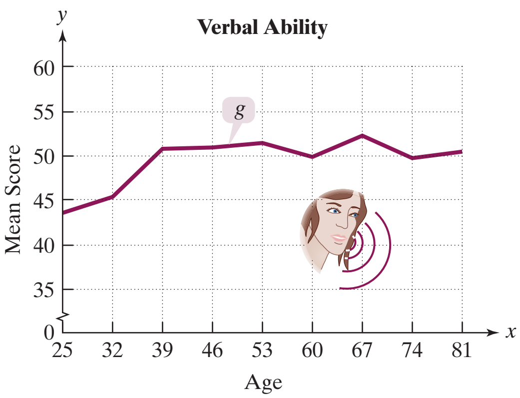
b. Would it be a good idea to move closer so that you actually reach $x = 3$? Describe the difference between what you feel for $\lim_{x\to 3} f(x)$ and $f(3)$.

68. You are riding along an expressway traveling x miles per hour. The function $f(x) = 0.015x^2 + x + 10$ describes the recommended safe distance, $f(x)$, in feet, between your car and other cars on the expressway. Use the values in the table below to find $\lim_{x \to 60} f(x)$. Describe what this means in terms of your car's speed and the recommended safe distance.

x	59.9	59.99	59.999 → ← 60.001	60.01	60.1
$f(x) = 0.015x^2 + x + 10$	123.72	123.972	123.997 → ← 124.003	124.028	124.28

Functions can be used to model changes in intellectual abilities over one's life span. The graphs of f and g show mean scores on standardized tests measuring spatial orientation and verbal ability, respectively, as a function of age. Use the graphs of f and g to solve Exercises 69–70.



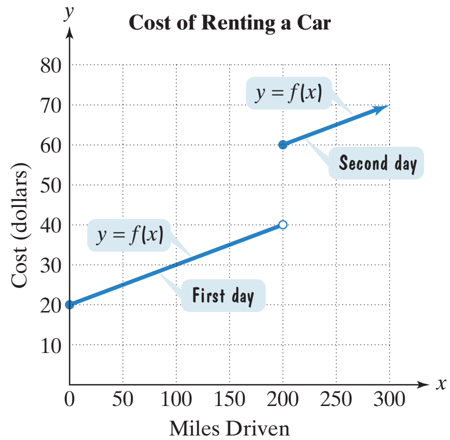


Source: Wade and Tavris, *Psychology Sixth Edition*, Prentice Hall, 2000

69. What mean score in spatial orientation is associated with a person whose age is close to 67? Use limit notation to express the answer.

70. What mean score in verbal ability is associated with a person whose age is close to 60? Use limit notation to express the answer.

71. You rent a car from a company that charges \$20 per day plus \$0.10 per mile. The car is driven 200 miles in the first day. The figure at the top of the next column shows the graph of the cost, $f(x)$, in dollars, as a function of the miles, x, that you drive the car.

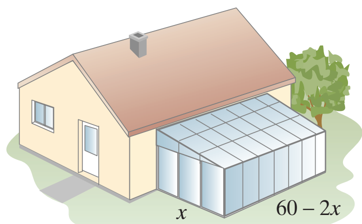


a. Find $\lim_{x \to 100} f(x)$. Interpret the limit, referring to miles driven and cost.

b. For the first day only, what is the rental cost approaching as the mileage gets closer to 200?

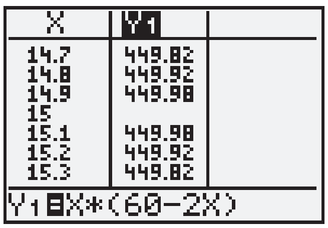
c. What is the cost to rent the car at the start of the second day?

72. You are building a greenhouse next to your house, as shown in the figure. Because the house will be used for one side of the enclosure, only three sides will need to be enclosed. You have 60 feet of fiberglass to enclose the three walls.

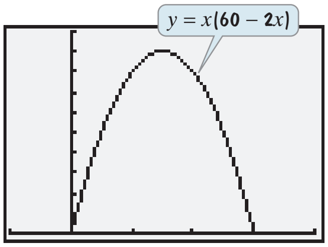


The function $f(x) = x(60 - 2x)$ describes the area of the greenhouse that you can enclose, $f(x)$, in square feet, if the width of the greenhouse is x feet.

a. Use the table shown to find $\lim_{x \to 15} f(x)$.



b. Use the graph shown to find $\lim_{x \to 15} f(x)$. Do you get the same limit as you did in part (a)? What information about the limit is shown by the graph that might not be obvious from the table?



$[-10, 40, 10]$ by $[0, 500, 50]$

Writing in Mathematics

73. Explain how to read $\lim_{x \to a} f(x) = L$.

74. What does the limit notation $\lim_{x \to a} f(x) = L$ mean?

75. Without showing the details, explain how to use a table to find $\lim_{x \to 4} x^2$.

76. Explain how a graph can be used to find a limit.

77. When we find $\lim_{x \to a} f(x)$, we do not care about the value of the function at $x = a$. Explain why this is so.

78. Explain how to read $\lim_{x \to a^-} f(x) = L$.

79. What does the limit notation $\lim_{x \to a^-} f(x) = L$ mean?

80. Explain how to read $\lim_{x \to a^+} f(x) = L$.

81. What does the limit notation $\lim_{x \to a^+} f(x) = L$ mean?

82. What does it mean if the limits in Exercises 79 and 81 are not both equal to the same number L?

Technology Exercises

83. Use the $\boxed{\text{TABLE}}$ feature of your graphing utility to verify any five of the limits that you found in Exercises 5–16.

84. Use the $\boxed{\text{ZOOM IN}}$ feature of your graphing utility to verify any five of the limits that you found in Exercises 33–46. Zoom in on the graph of the given function, f, near $x = a$ to verify each limit.

In Exercises 85–88, estimate $\lim_{x \to a} f(x)$ by using the $\boxed{\text{TABLE}}$ feature of your graphing utility to create a table of values. Then use the $\boxed{\text{ZOOM IN}}$ feature to zoom in on the graph of f near $x = a$ to justify or improve your estimate.

85. $\lim_{x \to 0} \dfrac{2^x - 1}{x}$

86. $\lim_{x \to 4} \dfrac{\ln x - \ln 4}{x - 4}$

87. $\lim_{x \to 1} \dfrac{x^{3/2} - 1}{x - 1}$

88. $\lim_{x \to 0} \dfrac{x^2}{1 - \cos 2x}$

Critical Thinking Exercises

Make Sense? *In Exercises 89–92, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

89. Limits indicate that a graph can get really close to values without actually reaching them.

90. I'm working with a function that is undefined at 5, so $\lim_{x \to 5} f(x)$ does not exist.

91. I'm working with a function that is undefined at 3, but defined at 2.99, 2.999, 2.9999, as well as at 3.01, 3.001, and 3.0001.

92. I'm working with a function for which $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, so I cannot draw the graph of the function near a without lifting my pencil off the paper.

93. Give an example of a function that is not defined at 2 for which $\lim_{x \to 2} f(x) = 5$.

94. Consider the function $f(x) = 3x + 2$. As x approaches 1, $f(x)$ approaches 5: $\lim_{x \to 1} f(x) = 5$. Find the values of x such that $f(x)$ is within 0.1 of 5 by solving

$$|f(x) - 5| < 0.1.$$

Then find the values of x such that $f(x)$ is within 0.01 of 5.

95. Find an estimate of 3^{π} ($\pi \approx 3.14159265$) by taking a sequence of rational numbers, $x_1, x_2, x_3, \ldots$ that approaches π. Obtain your estimate by evaluating $3^{x_1}, 3^{x_2}, 3^{x_3}, \ldots$.

Preview Exercises

Exercises 96–98 will help you prepare for the material covered in the next section.

96. **a.** Graph the piecewise function:

$$f(x) = \begin{cases} x^2 + 5 & \text{if } x < 2 \\ 3x + 1 & \text{if } x \geq 2. \end{cases}$$

b. Use your graph from part (a) to find each of the following limits, or indicate that the limit does not exist: $\lim_{x \to 2^-} f(x)$; $\lim_{x \to 2^+} f(x)$; $\lim_{x \to 2} f(x)$.

97. Simplify: $\dfrac{x^2 - x - 6}{x - 3}$.

98. Rationalize the numerator: $\dfrac{\sqrt{4 + x} - 2}{x}$.