Critical Thinking Exercises

Make Sense? *In Exercises 74–77, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

74. I evaluated a polynomial function f at 3 and obtained 7, so 7 must be $\lim_{x\to 3} f(x)$.

75. I'm working with functions f and g for which
$\lim_{x\to 4} f(x) = 0$, $\lim_{x\to 4} g(x) = -5$, and $\lim_{x\to 4} [f(x) - g(x)] = 5$.

76. I'm working with functions f and g for which

$$\lim_{x\to 4} f(x) = 0,\ \lim_{x\to 4} g(x) = -5, \text{ and } \lim_{x\to 4} \frac{f(x)}{g(x)} = 0.$$

77. I'm working with functions f and g for which

$$\lim_{x\to 4} f(x) = 0,\ \lim_{x\to 4} g(x) = -5, \text{ and } \lim_{x\to 4} \frac{g(x)}{f(x)} \neq 0.$$

In Exercises 78–79, find the indicated limit.

78. $\lim_{x\to 0} x\left(1 - \frac{1}{x}\right)$ **79.** $\lim_{x\to 4}\left(\frac{1}{x} - \frac{1}{4}\right)\left(\frac{1}{x-4}\right)$

In Exercises 80–81, find $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$.

80. $f(x) = x^2 + 2x - 3, a = 1$ **81.** $f(x) = \sqrt{x}, a = 1$

In Exercises 82–83, use properties of limits and the following limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0,$$
$$\lim_{x\to 0} \sin x = 0, \qquad \lim_{x\to 0} \cos x = 1$$

to find the indicated limit.

82. $\lim_{x\to 0} \frac{\tan x}{x}$ **83.** $\lim_{x\to 0} \frac{2\sin x + \cos x - 1}{3x}$

Group Exercises

84. In the next column is a list of ten common errors involving algebra, trigonometry, and limits that students frequently make in calculus. Group members should examine each error and describe the mistake. Where possible, correct each error. Finally, group members should offer suggestions for avoiding each error.

a. $(x + h)^3 - x^3 = x^3 + h^3 - x^3 = h^3$

b. $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$

c. $\frac{1}{a+b} = \frac{1}{a} + b$

d. $\sqrt{x+h} - \sqrt{x} = \sqrt{x} + \sqrt{h} - \sqrt{x} = \sqrt{h}$

e. $\frac{\sin 2x}{x} = \sin 2$

f. $\frac{a + bx}{a} = 1 + bx$

g. $\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \frac{1^3 - 1}{1 - 1} = \frac{0}{0} = 1$

h. $\sin(x + h) - \sin x = \sin x + \sin h - \sin x = \sin h$

i. $ax = bx$, so $a = b$

j. To find $\lim_{x\to 4} \frac{x^2 - 9}{x - 3}$, it is necessary to rewrite $\frac{x^2 - 9}{x - 3}$ by factoring $x^2 - 9$.

85. Research and present a group report about the history of the feud between Newton and Leibniz over who invented calculus. What other interests did these men have in addition to mathematics? What practical problems led them to the invention of calculus? What were their personalities like? Whose version established the notation and rules of calculus that we use today?

Preview Exercises

Exercises 86–88 will help you prepare for the material covered in the next section. In each exercise, use what occurs near 3 and at 3 to graph the function in an open interval about 3. (Graphs will vary.) Is it necessary to lift your pencil off the paper to obtain each graph? Explain your answer.

86. $\lim_{x\to 3} f(x) = 5; f(3) = 5$ **87.** $\lim_{x\to 3} f(x) = 5; f(3) = 6$

88. $\lim_{x\to 3^-} f(x) = 5;\ \lim_{x\to 3^+} f(x) = 6;\ f(3) = 5$

Section 11.3 Limits and Continuity

Objectives

1. Determine whether a function is continuous at a number.
2. Determine for what numbers a function is discontinuous.

Why you should not ski down discontinuous slopes

In everyday speech, a continuous process is one that goes on without interruption and without abrupt changes. In mathematics, a continuous function has much the same meaning. The graph of a continuous function does not have interrupting breaks, such as holes, gaps, or jumps. Thus, the graph of a continuous function can be drawn without lifting a pencil off the paper. In this section, you will learn how limits can be used to describe continuity.

1 Determine whether a function is continuous at a number.

Limits and Continuity

Figure 11.12 shows three graphs that cannot be drawn without lifting a pencil from the paper. In each case, there appears to be an interruption of the graph of f at $x = a$.

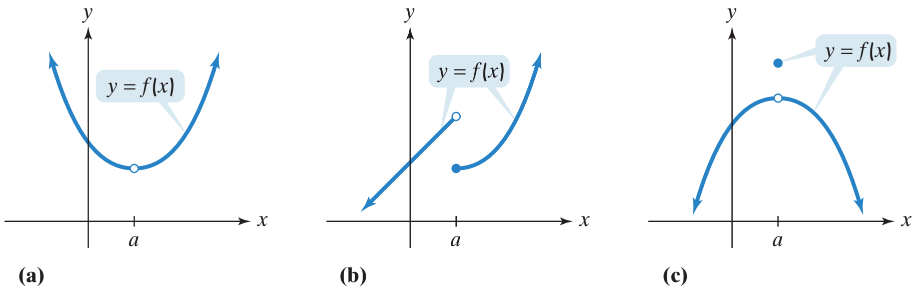


Figure 11.12 Each graph has an interruption at $x = a$.

Examine **Figure 11.12(a)**. The interruption occurs because the open dot indicates there is no point on the graph corresponding to $x = a$. This shows that $f(a)$ is not defined.

Now, examine **Figure 11.12(b)**. The closed blue dot at $x = a$ shows that $f(a)$ is defined. However, there is a jump at a. As x approaches a from the left, the values of f get closer to the y-coordinate of the point shown by the open dot. By contrast, as x approaches a from the right, the values of f get closer to the y-coordinate of the point shown by the closed dot. There is no single limit as x approaches a. The jump in the graph reflects the fact that $\lim_{x \to a} f(x)$ does not exist.

Finally, examine **Figure 11.12(c)**. The closed blue dot at $x = a$ shows that $f(a)$ is defined. Furthermore, as x approaches a from the left or from the right, the values of f get closer to the y-coordinate of the point shown by the open dot. Thus, $\lim_{x \to a} f(x)$ exists. However, there is still an interruption at a. Do you see why? The limit as x approaches a, $\lim_{x \to a} f(x)$, is the y-coordinate of the open dot. By contrast, the value of the function at a, $f(a)$, is the y-coordinate of the closed dot. The interruption in the graph reflects the fact that $\lim_{x \to a} f(x)$ and $f(a)$ are not equal.

We now provide a precise definition of what it means for a function to be continuous at a number. Notice how each part of this definition avoids the interruptions that occurred in **Figure 11.12**.

Definition of a Function Continuous at a Number

A function ***f*** **is continuous at *a*** when three conditions are satisfied.

1. f is defined at a; that is, a is in the domain of f, so that $f(a)$ is a real number.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

If f is not continuous at a, we say that f is **discontinuous at *a***. Each of the functions whose graph is shown in **Figure 11.12** is discontinuous at a.

EXAMPLE 1 Determining Whether a Function Is Continuous at a Number

Determine whether the function

$$f(x) = \frac{2x + 1}{2x^2 - x - 1}$$

is continuous: **a.** at 2; **b.** at 1.

Solution According to the definition, three conditions must be satisfied to have continuity at a.

a. To determine whether $f(x) = \dfrac{2x+1}{2x^2-x-1}$ is continuous at 2, we check the conditions for continuity with $a = 2$.

Condition 1 f is defined at a. Is $f(2)$ defined?

$$f(2) = \frac{2\cdot 2+1}{2\cdot 2^2-2-1} = \frac{4+1}{8-2-1} = \frac{5}{5} = 1$$

Because $f(2)$ is a real number, 1, $f(2)$ is defined.

Condition 2 $\lim\limits_{x\to a} f(x)$ exists. Does $\lim\limits_{x\to 2} f(x)$ exist?

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{2x+1}{2x^2-x-1} = \frac{\lim\limits_{x\to 2}(2x+1)}{\lim\limits_{x\to 2}(2x^2-x-1)}$$

$$= \frac{2\cdot 2+1}{2\cdot 2^2-2-1} = \frac{4+1}{8-2-1} = \frac{5}{5} = 1$$

Using properties of limits, we see that $\lim\limits_{x\to 2} f(x)$ exists.

Condition 3 $\lim\limits_{x\to a} f(x) = f(a)$ Does $\lim\limits_{x\to 2} f(x) = f(2)$? We found that $\lim\limits_{x\to 2} f(x) = 1$ and $f(2) = 1$. Thus, as x gets closer to 2, the corresponding values of $f(x)$ get closer to the function value at 2: $\lim\limits_{x\to 2} f(x) = f(2)$.

Because the three conditions are satisfied, we conclude that f is continuous at 2.

b. To determine whether $f(x) = \dfrac{2x+1}{2x^2-x-1}$ is continuous at 1, we check the conditions for continuity with $a = 1$.

Condition 1 f is defined at a. Is $f(1)$ defined? Factor the denominator of the function's equation:

$$f(x) = \frac{2x+1}{(x-1)(2x+1)}$$

Denominator is zero at $x = 1$.

Denominator is zero at $x = -\frac{1}{2}$.

Because division by zero is undefined, the domain of f is $\left\{x \mid x \neq 1, x \neq -\frac{1}{2}\right\}$. Thus, f is not defined at 1.

Because one of the three conditions is not satisfied, we conclude that f is not continuous at 1. Equivalently, we can say that f is discontinuous at 1.

The graph of $f(x) = \dfrac{2x+1}{2x^2-x-1}$ is shown in **Figure 11.13**. The graph verifies our work in Example 1. Can you see that f is continuous at 2? By contrast, it is not continuous at 1, where the graph has a vertical asymptote.

The graph in **Figure 11.13** also reveals a discontinuity at $-\frac{1}{2}$. The open dot indicates that there is no point on the graph corresponding to $x = -\frac{1}{2}$. Can you see what is happening as x approaches $-\frac{1}{2}$?

$$\lim_{x\to -\frac{1}{2}}\frac{2x+1}{2x^2-x-1} = \lim_{x\to -\frac{1}{2}}\frac{\cancel{2x+1}}{(x-1)\cancel{(2x+1)}} = \lim_{x\to -\frac{1}{2}}\frac{1}{x-1} = \frac{1}{-\frac{1}{2}-1} = -\frac{2}{3}$$

As x gets closer to $-\frac{1}{2}$, the graph of f gets closer to $-\frac{2}{3}$. Because f is not defined at $-\frac{1}{2}$, the graph has a hole at $\left(-\frac{1}{2}, -\frac{2}{3}\right)$. This is shown by the open dot in **Figure 11.13**.

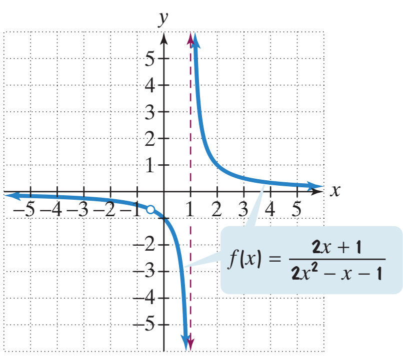


Figure 11.13 f is continuous at 2. It is not continuous at 1 or at $-\frac{1}{2}$.

Check Point 1 Determine whether the function

$$f(x) = \frac{x-2}{x^2-4}$$

is continuous: **a.** at 1; **b.** at 2.

2 Determine for what numbers a function is discontinuous.

Determining Where Functions Are Discontinuous

We have seen that the limit of a polynomial function as x approaches a is the polynomial function evaluated at a. Thus, if f is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$ for any number a. This means that **a polynomial function is continuous at every number**.

Many of the functions discussed throughout this book are continuous at every number in their domain. For example, rational functions are continuous at every number, except any at which they are not defined. At numbers that are not in the domain of a rational function, a hole in the graph or an asymptote appears. Exponential, logarithmic, sine, and cosine functions are continuous at every number in their domain. Like rational functions, the tangent, cotangent, secant, and cosecant functions are continuous at every number, except any at which they are not defined. At numbers that are not in the domain of these trigonometric functions, an asymptote occurs.

Study Tip

Most functions are always continuous at every number in their domain, including polynomial, rational, radical, exponential, logarithmic, and trigonometric functions. Most of the discontinuities you will encounter in calculus will be due to jumps in piecewise functions.

Example 2 illustrates how to determine where a piecewise function is discontinuous.

EXAMPLE 2 Determining Where a Piecewise Function Is Discontinuous

Determine for what numbers x, if any, the following function is discontinuous:

$$f(x) = \begin{cases} x + 2 & \text{if } x \le 0 \\ 2 & \text{if } 0 < x \le 1. \\ x^2 + 2 & \text{if } x > 1 \end{cases}$$

Solution First, let's determine whether each of the three pieces of f is continuous. The first piece, $f(x) = x + 2$, is a linear function; it is continuous at every number x. The second piece, $f(x) = 2$, a constant function, is continuous at every number x. And the third piece, $f(x) = x^2 + 2$, a polynomial function, is also continuous at every number x. Thus, these three functions, a linear function, a constant function, and a polynomial function, can be graphed without lifting a pencil from the paper. However, the pieces change at $x = 0$ and at $x = 1$. Is it necessary to lift a pencil from the paper when graphing f at these values? It appears that we must investigate continuity at 0 and at 1.

To determine whether the function is continuous at 0, we check the conditions for continuity with $a = 0$.

Condition 1 f is defined at a. Is $f(0)$ defined? Because $a = 0$, we use the first line of the piecewise function, where $x \le 0$.

$f(x) = x + 2$ This is the function's equation for $x \le 0$, which includes $x = 0$.

$f(0) = 0 + 2$ Replace x with 0.

$= 2$

Because $f(0)$ is a real number, 2, $f(0)$ is defined.

Condition 2 $\lim_{x \to a} f(x)$ exists. Does $\lim_{x \to 0} f(x)$ exist? To answer this question, we look at the values of $f(x)$ when x is close to 0. Let us investigate the left- and right-hand limits. If these limits are equal, then $\lim_{x \to 0} f(x)$ exists. To find $\lim_{x \to 0^-} f(x)$,

$$f(x) = \begin{cases} x + 2 & \text{if } x \le 0 \\ 2 & \text{if } 0 < x \le 1 \\ x^2 + 2 & \text{if } x > 1 \end{cases}$$

The given piecewise function (repeated)

the left-hand limit, we look at the values of $f(x)$ when x is close to 0, but less than 0. Because x is less than 0, we use the first line of the piecewise function, $f(x) = x + 2$ if $x \le 0$. Thus,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + 2) = 0 + 2 = 2.$$

To find $\lim_{x \to 0^+} f(x)$, the right-hand limit, we look at the values of $f(x)$ when x is close to 0, but greater than 0. Because x is greater than 0, we use the second line of the piecewise function, $f(x) = 2$ if $0 < x \le 1$. Thus,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 = 2.$$

Because the left- and right-hand limits are both equal to 2, $\lim_{x \to 0} f(x) = 2$. Thus, we see that $\lim_{x \to 0} f(x)$ exists.

Condition 3 $\boldsymbol{\lim_{x \to a} f(x) = f(a)}$ Does $\lim_{x \to 0} f(x) = f(0)$? We found that $\lim_{x \to 0} f(x) = 2$ and $f(0) = 2$. This means that as x gets closer to 0, the corresponding values of $f(x)$ get closer to the function value at 0: $\lim_{x \to 0} f(x) = f(0)$.

Because the three conditions are satisfied, we conclude that f is continuous at 0.

Now we must determine whether the function is continuous at 1, the other value of x where the pieces change. We check the conditions for continuity with $a = 1$.

Condition 1 $\boldsymbol{f}$ **is defined at** $\boldsymbol{a}$**.** Is $f(1)$ defined? Because $a = 1$, we use the second line of the piecewise function, where $0 < x \le 1$.

$f(x) = 2$ This is the function's equation for $0 < x \le 1$, which includes $x = 1$.

$f(1) = 2$ Replace x with 1.

Because $f(1)$ is a real number, 2, $f(1)$ is defined.

Condition 2 $\boldsymbol{\lim_{x \to a} f(x)}$ **exists.** Does $\lim_{x \to 1} f(x)$ exist? We investigate left- and right-hand limits as x approaches 1. To find $\lim_{x \to 1^-} f(x)$, the left-hand limit, we look at values of $f(x)$ when x is close to 1, but less than 1. Thus, we use the second line of the piecewise function, $f(x) = 2$ if $0 < x \le 1$. The left-hand limit is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2 = 2.$$

To find $\lim_{x \to 1^+} f(x)$, the right-hand limit, we look at values of $f(x)$ when x is close to 1, but greater than 1. Thus, we use the third line of the piecewise function, $f(x) = x^2 + 2$ if $x > 1$. The right-hand limit is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 2) = 1^2 + 2 = 3.$$

The left- and right-hand limits are not equal: $\lim_{x \to 1^-} f(x) = 2$ and $\lim_{x \to 1^+} f(x) = 3$. This means that $\lim_{x \to 1} f(x)$ does not exist.

Because one of the three conditions is not satisfied, we conclude that f is not continuous at 1.

In summary, the given function is discontinuous at 1 only. The graph of f, shown in **Figure 11.14**, illustrates this conclusion.

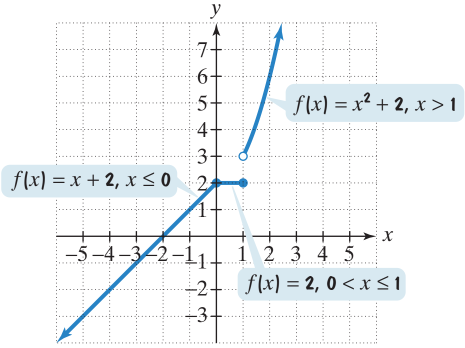


Figure 11.14 This piecewise function is continuous at 0, where pieces change, and discontinuous at 1, where pieces change.

Check Point 2 Determine for what numbers x, if any, the following function is discontinuous:

$$f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ x^2 + 1 & \text{if } 0 < x \le 2. \\ 7 - x & \text{if } x > 2 \end{cases}$$

Exercise Set 11.3

Practice Exercises

In Exercises 1–18, use the definition of continuity to determine whether f is continuous at a.

1. $f(x) = 2x + 5$
$a = 1$

2. $f(x) = 3x + 4$
$a = 1$

3. $f(x) = x^2 - 3x + 7$
$a = 4$

4. $f(x) = x^2 - 5x + 6$
$a = 4$

5. $f(x) = \dfrac{x^2 + 4}{x - 2}$
$a = 3$

6. $f(x) = \dfrac{x^2 + 6}{x - 5}$
$a = 6$

7. $f(x) = \dfrac{x + 5}{x - 5}$
$a = 5$

8. $f(x) = \dfrac{x + 7}{x - 7}$
$a = 7$

9. $f(x) = \dfrac{x - 5}{x + 5}$
$a = 5$

10. $f(x) = \dfrac{x - 7}{x + 7}$
$a = 7$

11. $f(x) = \dfrac{x^2 + 5x}{x^2 - 5x}$
$a = 0$

12. $f(x) = \dfrac{x^2 + 8x}{x^2 - 8x}$
$a = 0$

13. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$
$a = 2$

14. $f(x) = \begin{cases} \dfrac{x^2 - 36}{x - 6} & \text{if } x \neq 6 \\ 13 & \text{if } x = 6 \end{cases}$
$a = 6$

15. $f(x) = \begin{cases} x - 5 & \text{if } x \leq 0 \\ x^2 + x - 5 & \text{if } x > 0 \end{cases}$
$a = 0$

16. $f(x) = \begin{cases} x - 4 & \text{if } x \leq 0 \\ x^2 + x - 4 & \text{if } x > 0 \end{cases}$
$a = 0$

17. $f(x) = \begin{cases} 1 - x & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ x^2 - 1 & \text{if } x > 1 \end{cases}$
$a = 1$

18. $f(x) = \begin{cases} 2 - x & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ x^2 & \text{if } x > 1 \end{cases}$
$a = 1$

In Exercises 19–34, determine for what numbers, if any, the given function is discontinuous.

19. $f(x) = x^2 + 4x - 6$

20. $f(x) = x^2 + 8x - 10$

21. $f(x) = \dfrac{x + 1}{(x + 1)(x - 4)}$

22. $f(x) = \dfrac{x + 2}{(x + 2)(x - 5)}$

23. $f(x) = \dfrac{\sin x}{x}$

24. $f(x) = \dfrac{1 - \cos x}{x}$

25. $f(x) = \pi$

26. $f(x) = c$

27. $f(x) = \begin{cases} x - 1 & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$

28. $f(x) = \begin{cases} x - 2 & \text{if } x \leq 2 \\ x^2 - 1 & \text{if } x > 2 \end{cases}$

29. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

30. $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$

31. $f(x) = \begin{cases} x + 6 & \text{if } x \leq 0 \\ 6 & \text{if } 0 < x \leq 2 \\ x^2 + 1 & \text{if } x > 2 \end{cases}$

32. $f(x) = \begin{cases} x + 7 & \text{if } x \leq 0 \\ 7 & \text{if } 0 < x \leq 3 \\ x^2 - 1 & \text{if } x > 3 \end{cases}$

33. $f(x) = \begin{cases} 5x & \text{if } x < 4 \\ 21 & \text{if } x = 4 \\ x^2 + 4 & \text{if } x > 4 \end{cases}$

34. $f(x) = \begin{cases} 7x & \text{if } x < 6 \\ 41 & \text{if } x = 6 \\ x^2 + 6 & \text{if } x > 6 \end{cases}$

Practice Plus

In Exercises 35–38, graph each function. Then determine for what numbers, if any, the function is discontinuous.

35. $f(x) = \begin{cases} \sin x & \text{if } -\pi \leq x < 0 \\ -\sin x & \text{if } 0 \leq x < \pi \\ \cos x & \text{if } \pi \leq x \leq 2\pi \end{cases}$

36. $f(x) = \begin{cases} -\cos x & \text{if } -\pi \leq x < 0 \\ -\sin x & \text{if } 0 \leq x < \pi \\ \sin x & \text{if } \pi \leq x \leq 2\pi \end{cases}$

37. $f(x) = \begin{cases} -1 & \text{if } x \text{ is an integer.} \\ 1 & \text{if } x \text{ is not an integer.} \end{cases}$

38. $f(x) = \begin{cases} 2 & \text{if } x \text{ is an odd integer.} \\ -2 & \text{if } x \text{ is not an odd integer.} \end{cases}$

In Exercises 39–42, determine for what numbers, if any, the function is discontinuous. Construct a table to find any required limits.

39. $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

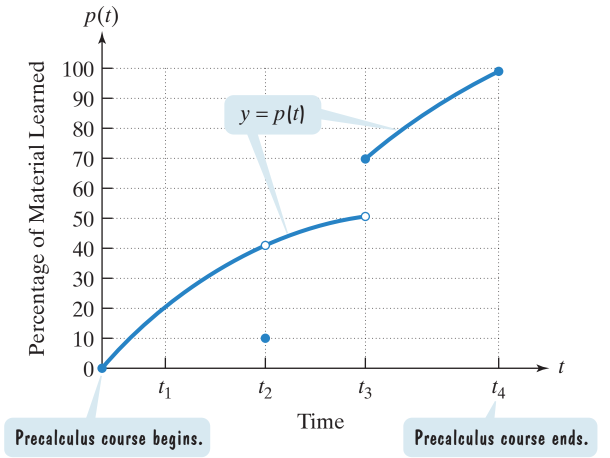
40. $f(x) = \begin{cases} \dfrac{\sin 3x}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$

41. $f(x) = \begin{cases} \dfrac{\cos x}{x - \frac{\pi}{2}} & \text{if } x \neq \dfrac{\pi}{2} \\ 1 & \text{if } x = \dfrac{\pi}{2} \end{cases}$

42. $f(x) = \begin{cases} \dfrac{\sin x}{x - \pi} & \text{if } x \neq \pi \\ 1 & \text{if } x = \pi \end{cases}$

Application Exercises

43. The graph represents the percentage of required topics in a precalculus course, $p(t)$, that a student learned at various times, t, throughout the course. At time t_2, the student panicked for an instant during an exam. At time t_3, after working on a set of cumulative review exercises, a big jump in understanding suddenly took place.



a. Find $\lim_{t \to t_1} p(t)$ and $p(t_1)$.

b. Is p continuous at t_1? Use the definition of continuity to explain your answer.

c. Find $\lim_{t \to t_2} p(t)$ and $p(t_2)$.

d. Is p continuous at t_2? Use the definition of continuity to explain your answer.

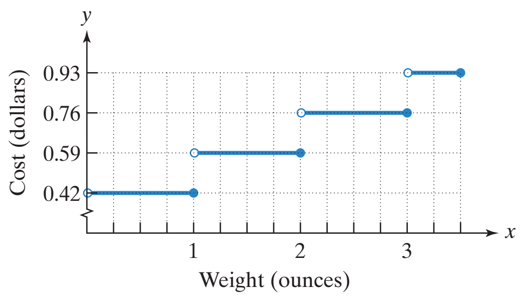
e. Find $\lim_{t \to t_3} p(t)$ and $p(t_3)$.

f. Is p continuous at t_3? Use the definition of continuity to explain your answer.

g. Find $\lim_{t \to t_4^-} p(t)$ and $p(t_4)$.

h. Explain the meaning of both the limit and the function value in part (g) in terms of the time in the course and the percentage of topics learned.

44. The figure shows the cost of mailing a first-class letter, $f(x)$, as a function of its weight, x, in ounces, for weights not exceeding 3.5 ounces.



Source: Lynn E. Baring, Postmaster, Inverness, CA

a. Find $\lim_{x \to 3^-} f(x)$.

b. Find $\lim_{x \to 3^+} f(x)$.

c. What can you conclude about $\lim_{x \to 3} f(x)$? How is this shown by the graph?

d. What aspect of costs for mailing a letter causes the graph to jump vertically by the same amount at its discontinuities?

45. The following piecewise function gives the tax owed, $T(x)$, by a single taxpayer in 2007 on a taxable income of x dollars.

$$T(x) = \begin{cases} 0.10x & \text{if} \quad 0 < x \le 7825 \\ 782.50 + 0.15(x - 7825) & \text{if} \quad 7825 < x \le 31{,}850 \\ 4386.25 + 0.25(x - 31{,}850) & \text{if} \quad 31{,}850 < x \le 77{,}100 \\ 15{,}698.75 + 0.28(x - 77{,}100) & \text{if} \quad 77{,}100 < x \le 160{,}850 \\ 39{,}148.75 + 0.33(x - 160{,}850) & \text{if} \quad 160{,}850 < x \le 349{,}700 \\ 101{,}469.25 + 0.35(x - 349{,}700) & \text{if} \quad x > 349{,}700. \end{cases}$$

a. Determine whether T is continuous at 7825.

b. Determine whether T is continuous at 31,850.

c. If T had discontinuities, use one of these discontinuities to describe a situation where it might be advantageous to earn less money in taxable income.

Writing in Mathematics

46. Explain how to determine whether a function is continuous at a number.

47. If a function is not defined at a, how is this shown on the function's graph?

48. If a function is defined at a, but $\lim_{x \to a} f(x)$ does not exist, how is this shown on the function's graph?

49. If a function is defined at a, $\lim_{x \to a} f(x)$ exists, but $\lim_{x \to a} f(x) \ne f(a)$, how is this shown on the function's graph?

50. In Exercises 43–44, functions that modeled learning in a precalculus course and the cost of mailing a letter had jumps in their graphs. Describe another situation that can be modeled by a function with discontinuities. What aspect of this situation causes the discontinuities?

51. Give two examples of the use of the word *continuous* in everyday English. Compare its use in your examples to its meaning in mathematics.

Technology Exercises

52. Use your graphing utility to graph any five of the functions in Exercises 1–18 and verify whether f is continuous at a.

53. Estimate $\lim_{x \to 0^+} (1 + x)^{1/x}$ by using the $\boxed{\text{TABLE}}$ feature of your graphing utility to create a table of values. Then use the $\boxed{\text{ZOOM IN}}$ feature to zoom in on the graph of f near and to the right of $x = 0$ to justify or improve your estimate.

Critical Thinking Exercises

Make Sense? *In Exercises 54–57, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

54. If $\lim_{x \to a} f(x) \ne f(a)$ and $\lim_{x \to a} f(x)$ exists, I can redefine $f(a)$ to make f continuous at a.

55. If $\lim_{x \to a^-} f(x) \ne f(a)$ and $\lim_{x \to a^-} f(x) \ne \lim_{x \to a^+} f(x)$, I can redefine $f(a)$ to make f continuous at a.

56. f and g are both continuous at a, although $f + g$ is not.

57. f and g are both continuous at a, although $\dfrac{f}{g}$ is not.

58. Define $f(x) = \dfrac{x^2 - 81}{x - 9}$ at $x = 9$ so that the function becomes continuous at 9.

59. Is it possible to define $f(x) = \dfrac{1}{x-9}$ at $x = 9$ so that the function becomes continuous at 9? How does this discontinuity differ from the discontinuity in Exercise 58?

60. For the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ Ax - 3 & \text{if } x \geq 1 \end{cases}$$

find A so that the function is continuous at 1.

Group Exercise

61. In this exercise, the group will define three piecewise functions. Each function should have three pieces and two values of x at which the pieces change.

a. Define and graph a piecewise function that is continuous at both values of x where the pieces change.

b. Define and graph a piecewise function that is continuous at one value of x where the pieces change and discontinuous at the other value of x where the pieces change.

c. Define and graph a piecewise function that is discontinuous at both values of x where the pieces change.

At the end of the activity, group members should turn in the functions and their graphs. Do not use any of the piecewise functions or graphs that appear anywhere in this book.

Preview Exercises

Exercises 62–64 will help you prepare for the material covered in the next section. In each exercise, find the indicated difference quotient and simplify.

62. If $f(x) = x^2 + x$, find $\dfrac{f(2+h) - f(2)}{h}$.

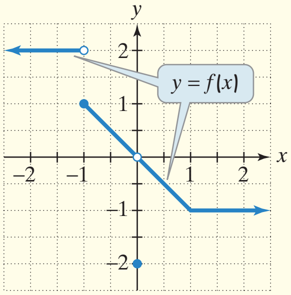
63. If $f(x) = x^3$, find $\dfrac{f(x+h) - f(x)}{h}$.

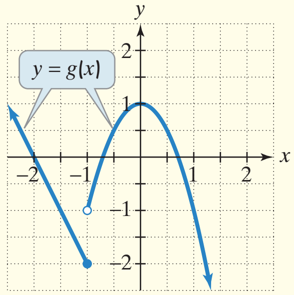
64. If $s(t) = -16t^2 + 48t + 160$, find $\dfrac{s(a+h) - s(a)}{h}$.

Chapter 11 Mid-Chapter Check Point

What you know: We learned that $\lim_{x \to a} f(x) = L$ means that as x gets closer to a, but remains unequal to a, the corresponding values of $f(x)$ get closer to L. We found limits using tables, graphs, and properties of limits. The quotient property for limits did not apply to fractional expressions in which the limit of the denominator is zero. In these cases, rewriting the expression using factoring or rationalizing the numerator or denominator was helpful before finding the limit. We saw that if the left-hand limit, $\lim_{x \to a^-} f(x)$, (x approaches a from the left) is not equal to the right-hand limit, $\lim_{x \to a^+} f(x)$, (x approaches a from the right), then $\lim_{x \to a} f(x)$ does not exist. Finally, we defined continuity in terms of limits. A function f is continuous at a when f is defined at a, $\lim_{x \to a} f(x)$ exists, and $\lim_{x \to a} f(x) = f(a)$. If f is not continuous at a, we say that f is discontinuous at a.

In Exercises 1–7, use the graphs of f and g to find the indicated limit or function value, or state that the limit or function value does not exist.





1. $\lim_{x \to -1^-} f(x)$

2. $\lim_{x \to -1^+} f(x)$

3. $\lim_{x \to -1} f(x)$

4. $\lim_{x \to 1} [f(x) + g(x)]$

5. $\lim_{x \to 0} [f(x) - g(x)]$

6. $(f - g)(0)$

7. $\lim_{x \to 1} \sqrt{10 + f(x)}$

8. Use the graph of f shown above to determine for what numbers the function is discontinuous. Then use the definition of continuity to verify each discontinuity.

In Exercises 9–11, use the table to find the indicated limit.

x	−0.03	−0.02	−0.01	−0.007	0.007	0.01	0.02	0.03
$f(x) = \dfrac{\sin x}{2x^2 - x}$	−0.9433	−0.9615	−0.9804	−0.9862	−1.014	−1.02	−1.042	−1.064
$g(x) = \dfrac{e^x - \tan x}{\cos^2 x}$	1.0014	1.0006	1.0002	1.0001	1.0001	1.0001	1.0006	1.0013