(In Exercises 9–11, be sure to refer to the table at the bottom of the previous page.)

9. $\lim_{x \to 0} f(x)$ **10.** $\lim_{x \to 0} g(x)$ **11.** $\lim_{x \to 0} \dfrac{4g(x)}{[f(x)]^2}$

In Exercises 12–17, find the limits.

12. $\lim_{x \to -2} (x^3 - x + 5)$

13. $\lim_{x \to 3} \sqrt{x^2 - 3x + 4}$

14. $\lim_{x \to 5} \dfrac{2x^2 - x + 4}{x - 1}$

15. $\lim_{x \to 5} \dfrac{2x^2 - 7x - 15}{x - 5}$

16. $\lim_{x \to 0} \dfrac{\sqrt{x^2 + 9} - 3}{x^2}$

17. $\lim_{x \to 0} \dfrac{\dfrac{1}{x + 10} - \dfrac{1}{x}}{x}$

In Exercises 18–19, a piecewise function is given. Use the function to find the indicated limit, or state that the limit does not exist.

18. $f(x) = \begin{cases} 9 - 2x & \text{if } x < 4 \\ \sqrt{x - 4} & \text{if } x \geq 4 \end{cases}$

a. $\lim_{x \to 4^-} f(x)$ **b.** $\lim_{x \to 4^+} f(x)$ **c.** $\lim_{x \to 4} f(x)$

19. $f(x) = \begin{cases} \dfrac{x^4 - 16}{x - 2} & \text{if } x \neq 2 \\ 32 & \text{if } x = 2 \end{cases}$

a. $\lim_{x \to 2^-} f(x)$ **b.** $\lim_{x \to 2^+} f(x)$ **c.** $\lim_{x \to 2} f(x)$

In Exercises 20–21, use the definition of continuity to determine whether f is continuous at a.

20. $f(x) = \begin{cases} \sqrt{3 - x} & \text{if } x \leq 3 \\ x^2 - 3x & \text{if } x > 3 \end{cases}$

$a = 3$

21. $f(x) = \begin{cases} \dfrac{(x + 3)^2 - 9}{x} & \text{if } x \neq 0 \\ 6 & \text{if } x = 0 \end{cases}$

$a = 0$

22 Determine for what numbers, if any, the following function is discontinuous:

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{if } x < -1 \\ 2x & \text{if } -1 \leq x \leq 5. \\ 3x - 4 & \text{if } x > 5 \end{cases}$$

Section 11.4 Introduction to Derivatives

Objectives

1. Find slopes and equations of tangent lines.
2. Find the derivative of a function.
3. Find average and instantaneous rates of change.
4. Find instantaneous velocity.

Things change over time and most changes occur at uneven rates. This is illustrated in the chapter opener (page 1037) with a sequence of photos of a young boy transforming into an adult. What does calculus have to say about this radical transformation?

In this section, we will see how calculus allows motion and change to be analyzed by "freezing the frame" of a continuously changing process, instant by instant. For example, **Figure 11.15** shows a male's changing height over intervals of time. Over the period of time from P to D, his average rate of growth is his change in height—that is, his height at time D minus his height at time P—divided by the change in time from P to D.

The lines PD, PC, PB, and PA shown in **Figure 11.15** have slopes that show the man's average growth rates for successively shorter periods of time. Calculus makes these time frames so small that their limit approaches a single point—that is, a single instant in time. This point is shown as point P in **Figure 11.15**. The slope of the line that touches the graph at P gives the man's growth rate at one instant in time, P.

Keep this informal discussion of this man and his growth rate in mind as you read this section. We begin with the calculus that describes the slope of the line that touches the graph in **Figure 11.15** at P.

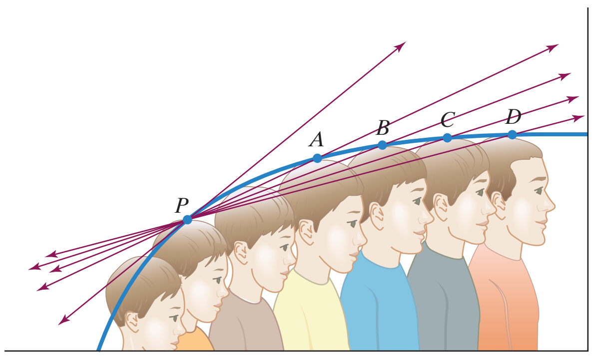


Figure 11.15

1 Find slopes and equations of tangent lines.

Slopes and Equations of Tangent Lines

In Chapter 1, we saw that if the graph of a function is not a straight line, the **average rate of change** between any two points is the slope of the line containing the two points. We called this line a **secant line**.

Figure 11.16 shows the graph of a male's height, in inches, as a function of his age, in years. Two points on the graph are labeled: (13, 57) and (18, 76). At age 13, this person was 57 inches tall, and at age 18, he was 76 inches tall. The slope of the secant line containing these two point is

$$\frac{76-57}{18-13} = \frac{19}{5} = 3\frac{4}{5}.$$

Slope is the change in the y-coordinates divided by the change in the x-coordinates.

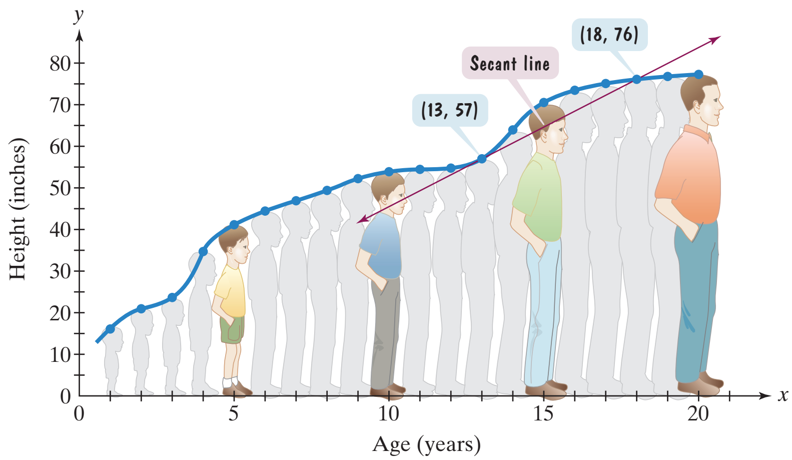


Figure 11.16 Height as a function of age

The man's average rate of change, or average growth rate, from 13 to 18 was $3\frac{4}{5}$ inches per year.

How can we find this person's growth rate at the instant when he was 13? We can find this *instantaneous rate of change* by repeating the computation of slope from 13 to 17, then from 13 to 16, then from 13 to 15, again from 13 to 14, again from 13 to $13\frac{1}{2}$, and once again from 13 to 13.01. What limit is approached by these computations as the shrinking interval of time gets closer and closer to the instant when your cousin was 13?

We answer these questions by considering the graph of any function f, shown in **Figure 11.17**. We need to find the slope, or steepness, of this curve at the point $P = (a, f(a))$. This slope will reveal the function's instantaneous rate of change at a. We begin by choosing a second point, Q, whose x-coordinate is $a + h$, where $h \neq 0$. The point $Q = (a + h, f(a + h))$ is shown in **Figure 11.17**.

How do we find the average rate of change of f between points P and Q? We find the slope of the secant line, the line containing P and Q.

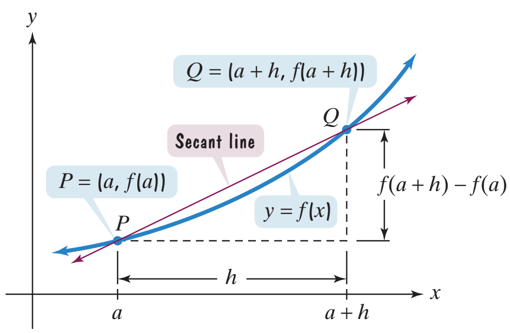


Figure 11.17 Finding the average rate of change, or the slope of the secant line

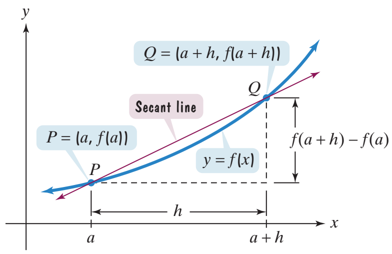


Figure 11.17 (repeated)

Slope of secant line

$$= \frac{f(a+h) - f(a)}{a + h - a}$$ Slope is the change in y-coordinates, $f(a + h) - f(a)$, divided by the change in x-coordinates, $(a + h) - a$.

$$= \frac{f(a+h) - f(a)}{h}$$ Simplify.

Do you recognize this expression as the difference quotient presented in Chapter 1? We will make use of this expression and our understanding of limits to find the slope of a graph at a specific point.

What happens if the distance labeled h in **Figure 11.17** approaches 0? The value of the x-coordinate of point Q, $a + h$, will get closer and closer to a. Can you see that a is the x-coordinate of point P? Thus, as h approaches 0, point Q approaches point P. Examine **Figure 11.18** to see how we visualize the changing position of point Q.

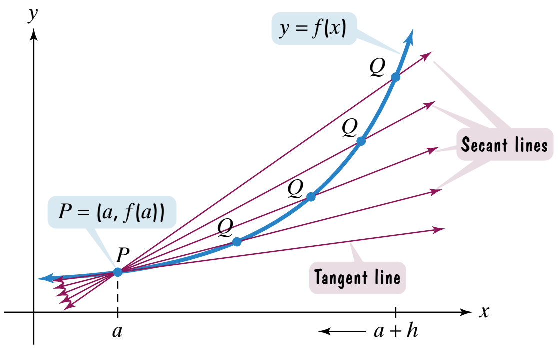


Figure 11.18 As Q approaches P, the succession of secant lines approaches the tangent line.

Figure 11.18 also shows how the secant line between points P and Q changes as h approaches 0. Note how the position of the secant line changes as the position of Q changes. The secant line between point P and point Q approaches the red line that touches the graph of f at point P. This limiting position of the secant line is called the **tangent line** to the graph of f at the point $P = (a, f(a))$.

According to our earlier derivation, the slope of each secant line in **Figure 11.18** is

$$\frac{f(a+h) - f(a)}{h}.$$ This difference quotient is also the average rate of change of f from $x_1 = a$ to $x_2 = a + h$.

As h approaches 0, this slope approaches the slope of the tangent line to the curve at $(a, f(a))$. Thus, the slope of the tangent line to the curve at $(a, f(a))$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This limit also represents the **instantaneous rate of change of *f* with respect to *x* at *a***.

Slope of the Tangent Line to a Curve at a Point

The **slope of the tangent line** to the graph of a function $y = f(x)$ at $(a, f(a))$ is given by

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists. This limit also describes

- the **slope of the graph of *f* at $(a, f(a))$**.
- the **instantaneous rate of change of *f* with respect to *x* at *a***.

EXAMPLE 1 Finding the Slope of a Tangent Line

Find the slope of the tangent line to the graph of $f(x) = x^2 + x$ at $(2, 6)$.

Solution The slope of the tangent line at $(a, f(a))$ is

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We use this formula to find the slope of the tangent line at the given point. Because we are finding the slope of the tangent line at $(2, 6)$, we know that $a = 2$.

$$m_{\tan} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$ Because $a = 2$, substitute 2 into the formula for each occurrence of a.

$$= \lim_{h \to 0} \frac{[(2+h)^2 + (2+h)] - [2^2 + 2]}{h}$$ To find $f(2 + h)$, replace x in $f(x) = x^2 + x$ with $2 + h$. To find $f(2)$ replace x with 2.

$$= \lim_{h \to 0} \frac{[4 + 4h + h^2 + 2 + h] - 6}{h}$$ Square $2 + h$ using $(A + B)^2 = A^2 + 2AB + B^2$.

$$= \lim_{h \to 0} \frac{h^2 + 5h}{h}$$ Combine like terms in the numerator.

$$= \lim_{h \to 0} \frac{h(h+5)}{h}$$ Factor the numerator.

$$= \lim_{h \to 0} (h + 5)$$ Divide both the numerator and denominator by h. This is permitted because h approaches 0, but $h \neq 0$.

$$= 0 + 5$$ Use limit properties.

$$= 5$$

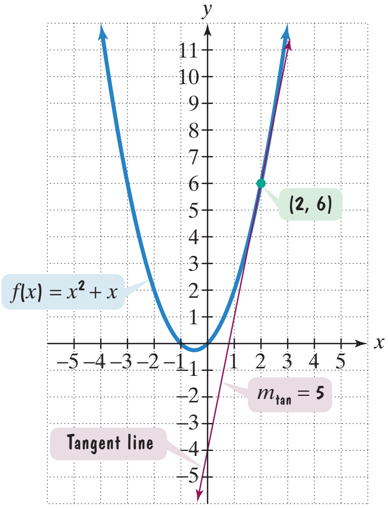


Figure 11.19

Thus, the slope of the tangent line to the graph of $f(x) = x^2 + x$ at $(2, 6)$ is 5. This is shown in **Figure 11.19**. We also say that the slope of the graph of $f(x) = x^2 + x$ at $(2, 6)$ is 5.

Check Point 1 Find the slope of the tangent line to the graph of $f(x) = x^2 - x$ at $(4, 12)$.

Technology

Graphing utilities with a DRAW TANGENT feature will draw tangent lines to curves and display their slope-intercept equations. **Figure 11.20** shows the tangent line to the graph of $y = x^2 + x$ at the point whose x-coordinate is 2. Also displayed is the slope-intercept equation of the tangent line, $y = 5x - 4$.

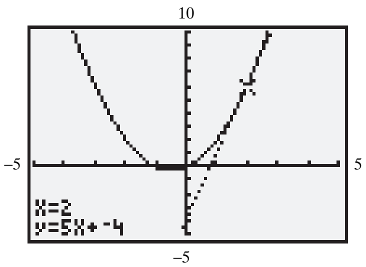


Figure 11.20

In Example 1, we found the *slope* of the tangent line shown in **Figure 11.19**. We can find an *equation* of this line using the point-slope form of the equation of a line

$$y - y_1 = m(x - x_1).$$

The tangent line passes through $(2, 6)$: $x_1 = 2$ and $y_1 = 6$. The slope of the tangent line is 5: $m = 5$. The point-slope equation of the tangent line is

$$y - 6 = 5(x - 2).$$

We can solve for y and express the equation of the tangent line in slope-intercept form: $y = mx + b$. The slope-intercept equation of the tangent line is

$$y - 6 = 5x - 10$$ Apply the distributive property.

$$y = 5x - 4.$$ Add 6 to both sides and write in slope-intercept form.

EXAMPLE 2 Finding the Slope-Intercept Equation of a Tangent Line

Find the slope-intercept equation of the tangent line to the graph of $f(x) = \sqrt{x}$ at $(4, 2)$.

Solution We begin by finding the slope of the tangent line to the graph of $f(x) = \sqrt{x}$ at $(4, 2)$.

$$m_{\tan} = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

The slope of the tangent line at $(a, f(a))$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

$$= \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$$

To find $f(4+h)$, replace x in $f(x) = \sqrt{x}$ with $4 + h$. $f(4) = \sqrt{4} = 2$

$$= \lim_{h \to 0} \left[\frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right]$$

Rationalize the numerator.

$$= \lim_{h \to 0} \frac{4 + h - 4}{h\left(\sqrt{4+h} + 2\right)}$$

Multiply the numerators.

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{4+h} + 2\right)}$$

Simplify the numerator.

$$= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2}$$

Divide both the numerator and the denominator by h. This is permitted because h approaches 0, but $h \neq 0$.

$$= \frac{1}{\sqrt{4+0} + 2}$$

Use limit properties.

$$= \frac{1}{4}$$

Now that we have the slope of the tangent line, we can write the slope-intercept equation. Begin with the point-slope form

$$y - y_1 = m(x - x_1).$$

The tangent line is given to pass through (4, 2): $x_1 = 4$ and $y_1 = 2$. We found the slope of the tangent line to be $\frac{1}{4}$: $m = \frac{1}{4}$. The point-slope equation of the tangent line is

$$y - 2 = \tfrac{1}{4}(x - 4).$$

Solving for y, we obtain the slope-intercept equation of the tangent line.

$$y - 2 = \tfrac{1}{4}x - 1$$

Apply the distributive property.

$$y = \tfrac{1}{4}x + 1$$

Add 2 to both sides. This is the slope-intercept form, $y = mx + b$, of the equation.

The slope-intercept equation of the tangent line to the graph of $f(x) = \sqrt{x}$ at $(4, 2)$ is $y = \frac{1}{4}x + 1$. **Figure 11.21** shows the graph of f and the tangent line.

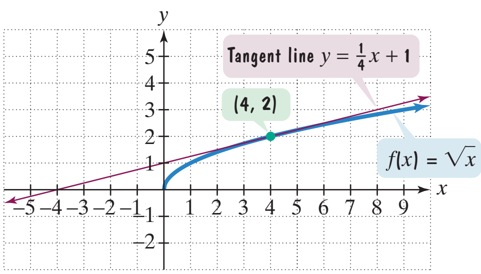


Figure 11.21

Check Point 2 Find the slope-intercept equation of the tangent line to the graph of $f(x) = \sqrt{x}$ at $(1, 1)$.

2 Find the derivative of a function.

The Derivative of a Function

In Examples 1 and 2, we found the slope of a tangent line to the graph of f at $(a, f(a))$, where a was a specific number. We can also find the slope of a tangent line at $(x, f(x))$, where x can represent any number in the domain of f'. The resulting function is called the *derivative of f at x*.

Definition of the Derivative of a Function

Let $y = f(x)$ denote a function f. The **derivative of *f* at *x***, denoted by $f'(x)$, read "f prime of x," is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that this limit exists. The derivative of a function f gives the slope of f for any value of x in the domain of f'.

By evaluating the derivative, you can compute the slopes of various tangent lines to the graph of a function. Thus, the derivative gives you a way to analyze your moving world by revealing a function's instantaneous rate of change at any moment.

The Seeds of Change

Every shape that's born bears in its womb the seeds of change.

—Ovid (Roman poet)

Figure 11.22 shows a graph involving change, namely a male's height as a function of his age. The derivative of this function provides a formula for the slope of the tangent line to the function's graph at any point. The figure shows four tangent lines. The derivative of this function would reveal that the tangent line with the greatest slope touches the curve somewhere between $x = 3$ and $x = 4$. Thus, the instantaneous rate of change in the boy's growth is greatest at some moment in time between the ages of 3 and 4.

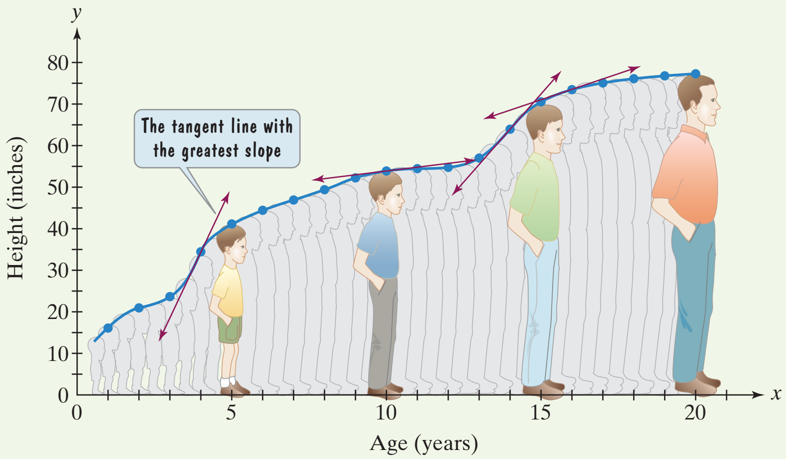


Figure 11.22 Analyzing continuous change at an instant

EXAMPLE 3 Finding the Derivative of a Function

a. Find the derivative of $f(x) = x^2 + 3x$ at x. That is, find $f'(x)$.

b. Find the slope of the tangent line to the graph of $f(x) = x^2 + 3x$ at $x = -2$ and at $x = -\frac{3}{2}$.

Solution

a. We use the definition of the derivative of f at x to find the derivative of the given function.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ Use the definition of the derivative.

$$= \lim_{h\to 0} \frac{[(x+h)^2 + 3(x+h)] - (x^2 + 3x)}{h}$$ To find $f(x+h)$, replace x in $f(x) = x^2 + 3x$ with $x + h$.

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h}$$ Perform the indicated operations in the numerator.

$$= \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h}$$ Simplify the numerator: $x^2 - x^2 = 0$ and $3x - 3x = 0$.

$$= \lim_{h\to 0} \frac{h(2x + h + 3)}{h}$$ Factor the numerator.

$$= \lim_{h\to 0} (2x + h + 3)$$ Divide the numerator and the denominator by h.

$$= 2x + 0 + 3$$ Use limit properties. As h approaches 0, only the term containing h is affected.

$$= 2x + 3$$

The derivative of $f(x) = x^2 + 3x$ is

$$f'(x) = 2x + 3.$$

b. The derivative gives the slope of the tangent line at any point. Thus, to find the slope of the tangent line to the graph of $f(x) = x^2 + 3x$ at $x = -2$, evaluate the derivative at -2. Similarly, to find the slope of the tangent line at $x = -\frac{3}{2}$, evaluate the derivative at $-\frac{3}{2}$.

$$f'(x) = 2x + 3$$
$$f'(-2) = 2(-2) + 3 = -4 + 3 = -1$$
$$f'\left(-\tfrac{3}{2}\right) = 2\left(-\tfrac{3}{2}\right) + 3 = -3 + 3 = 0$$

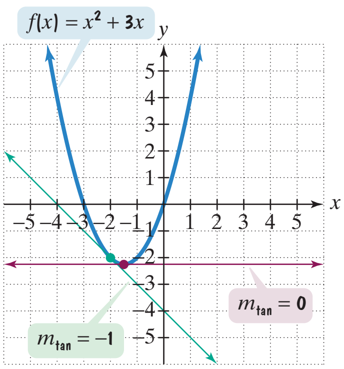


Figure 11.23 Two tangent lines to the graph of $f(x) = x^2 + 3x$ and their slopes

Figure 11.23 shows the graph of $f(x) = x^2 + 3x$ and tangent lines at $x = -2$ and $x = -\frac{3}{2}$. The slope of the decreasing green tangent line at $x = -2$ is -1. The slope of the horizontal red tangent line at $x = -\frac{3}{2}$ is 0.

Check Point 3

a. Find the derivative of $f(x) = x^2 - 5x$ at x. That is, find $f'(x)$.

b. Find the slope of the tangent line to the graph of $f(x) = x^2 - 5x$ at $x = -1$ and at $x = 3$.

3 Find average and instantaneous rates of change.

Applications of the Derivative

Many applications of the derivative involve analyzing change by determining a function's instantaneous rate of change at any moment. How do we use the derivative of a function to reveal such changes? We know that the derivative of f at x is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Thus, the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Do you recognize this limit? It describes the instantaneous rate of change of f with respect to x at a.

Average and Instantaneous Rates of Change

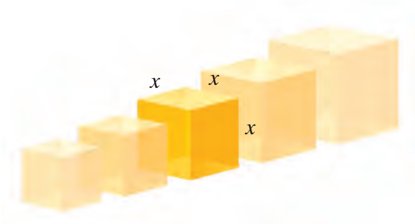
Average Rate of Change The **average rate of change of f from $x = a$ to $x = a + h$** is given by the difference quotient

$$\frac{f(a + h) - f(a)}{h}.$$

Instantaneous Rate of Change The **instantaneous rate of change of f with respect to x at a** is the derivative of f at a:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

EXAMPLE 4 Finding Average and Instantaneous Rates of Change



The function $f(x) = x^3$ describes the volume of a cube, $f(x)$, in cubic inches, whose length, width, and height each measure x inches. If x is changing,

a. Find the average rate of change of the volume with respect to x as x changes from 5 inches to 5.1 inches and from 5 inches to 5.01 inches.

b. Find the instantaneous rate of change of the volume with respect to x at the moment when $x = 5$ inches.

Solution

a. As x changes from 5 to 5.1, $a = 5$ and $h = 0.1$. The average rate of change of the volume with respect to x as x changes from 5 to 5.1 is determined as follows.

$$\frac{f(a + h) - f(a)}{h}$$ The difference quotient gives the average rate of change from a to $a + h$.

$$= \frac{f(5 + 0.1) - f(5)}{0.1}$$ This is the average rate of change from 5 to 5.1.

$$= \frac{f(5.1) - f(5)}{0.1}$$ Simplify.

$$= \frac{5.1^3 - 5^3}{0.1}$$ Use $f(x) = x^3$ and substitute 5.1 and 5, respectively, for x.

$$= 76.51$$

The average rate of change in the volume is 76.51 cubic inches per inch as x changes from 5 to 5.1 inches.

As x changes from 5 to 5.01, $a = 5$ and $h = 0.01$. The average rate of change of the volume with respect to x as x changes from 5 to 5.01 is determined as follows.

$$\frac{f(a + h) - f(a)}{h}$$ The difference quotient gives the average rate of change from a to $a + h$.

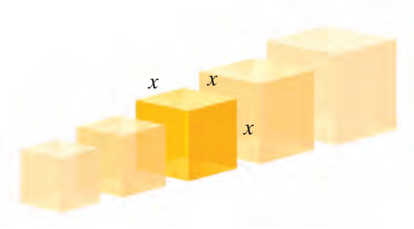
$$= \frac{f(5 + 0.01) - f(5)}{0.01}$$ This is the average rate of change from 5 to 5.01.

$$= \frac{f(5.01) - f(5)}{0.01}$$ Simplify.

$$= \frac{5.01^3 - 5^3}{0.01}$$ Use $f(x) = x^3$ and substitute 5.01 and 5, respectively, for x.

$$= 75.1501$$

The average rate of change in the volume is 75.1501 cubic inches per inch as x changes from 5 to 5.01 inches.



b. Instantaneous rates of change are given by the derivative. The derivative of f at a, $f'(a)$, is the instantaneous rate of change of f with respect to x at a. We must find the instantaneous rate of change of the volume with respect to x at the moment when $x = 5$ inches. This means that we must find $f'(5)$. We find $f'(5)$ by first finding $f'(x)$, the derivative, and then evaluating f' at 5.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ Use the definition of the derivative.

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$ To find $f(x + h)$, replace x in $f(x) = x^3$ with $x + h$.

$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$ Use the Binomial Theorem to cube $x + h$.

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$ Simplify the numerator: $x^3 - x^3 = 0$.

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$ Factor the numerator.

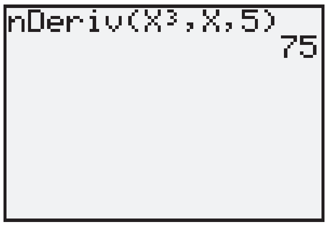
$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$ Divide the numerator and the denominator by h.

$$= 3x^2 + 3x \cdot 0 + 0^2$$ Use limit properties. As h approaches 0, only terms containing h are affected.

$$= 3x^2$$

Technology

Graphing utilities have a feature that gives (or approximates) the derivative of a function evaluated at any number. Consult your manual for details. The screen below verifies that if $f(x) = x^3$, then $f'(5) = 75$.



The derivative of $f(x) = x^3$ is $f'(x) = 3x^2$. To find the instantaneous change of f at 5, evaluate the derivative at 5.

$$f'(x) = 3x^2$$
$$f'(5) = 3 \cdot 5^2 = 75$$

The instantaneous rate of change of the volume with respect to x at the moment when $x = 5$ inches is 75 cubic inches per inch. Notice how the average rates of change that we computed in part (a), 76.51 and 75.1501, are approaching the instantaneous rate of change, 75.

Check Point 4 Use the function in Example 4, $f(x) = x^3$, to find each of the following:

a. the average rate of change of the volume with respect to x as x changes from 4 inches to 4.1 inches and from 4 inches to 4.01 inches.

b. the instantaneous rate of change of the volume with respect to x at the moment when $x = 4$ inches.

4 Find instantaneous velocity.

The ideas of calculus are frequently applied to position functions that express an object's position, $s(t)$, in terms of time, t. In the time interval from $t = a$ to $t = a + h$, the change in the object's position is

$$s(a+h) - s(a).$$

The **average velocity** over this time interval is

$$\frac{s(a+h) - s(a)}{h}.$$

The numerator is the change in position.

The denominator is the change in time from $t = a$ to $t = a + h$.

Now suppose that we compute the average velocities over shorter and shorter time intervals $[a, a + h]$. This means that we let h approach 0. As in our previous discussion, we define the *instantaneous velocity* at time $t = a$ to be the limit of these average velocities. This limit is the derivative of s at a.

Instantaneous Velocity

Suppose that a function expresses an object's position, $s(t)$, in terms of time, t. The **instantaneous velocity** of the object at time $t = a$ is

$$s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}.$$

Instantaneous velocity at time a is also called **velocity** at time a.

EXAMPLE 5 Finding Instantaneous Velocity

A ball is thrown straight up from a rooftop 160 feet high with an initial velocity of 48 feet per second. The function

$$s(t) = -16t^2 + 48t + 160$$

describes the ball's height above the ground, $s(t)$, in feet, t seconds after it is thrown. The ball misses the rooftop on its way down and eventually strikes the ground.

a. What is the instantaneous velocity of the ball 2 seconds after it is thrown?

b. What is the instantaneous velocity of the ball when it hits the ground?

Solution Instantaneous velocity is given by the derivative of a function that expresses an object's position, $s(t)$, in terms of time, t. The instantaneous velocity of the ball at a seconds is $s'(a)$.

$$s'(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$ This derivative describes instantaneous velocity at time a.

To find $s(a + h)$, replace t in $s(t) = -16t^2 + 48t + 160$ with $a + h$. To find $s(a)$, replace t with a. Thus,

$$s'(a) = \lim_{h \to 0} \frac{-16(a+h)^2 + 48(a+h) + 160 - (-16a^2 + 48a + 160)}{h}$$

Take a few minutes to simplify the numerator of the difference quotient and factor out h. You should obtain

$$s'(a) = \lim_{h \to 0} \frac{\cancel{h}(-32a - 16h + 48)}{\cancel{h}} = -32a - 16 \cdot 0 + 48 = -32a + 48.$$

The instantaneous velocity of the ball at a seconds is

$$s'(a) = -32a + 48.$$

a. The instantaneous velocity of the ball at 2 seconds is found by replacing a with 2.

$$s'(2) = -32 \cdot 2 + 48 = -64 + 48 = -16$$

Two seconds after the ball is thrown, its instantaneous velocity is -16 feet per second. The negative sign indicates that the ball is moving downward when $t = 2$ seconds.

b. To find the instantaneous velocity of the ball when it hits the ground, we need to know how many seconds elapse between the time the ball is thrown from the rooftop and the time it hits the ground. The ball hits the ground when $s(t)$, its height above the ground, is 0. Thus, we set $s(t)$ equal to 0.

$$-16t^2 + 48t + 160 = 0$$ Set $s(t) = 0$.

$$-16(t^2 - 3t - 10) = 0$$ Factor out -16.

$$-16(t-5)(t+2) = 0$$ Factor completely.

$$t - 5 = 0 \qquad t + 2 = 0$$ Set each variable factor equal to 0.

$$t = 5 \qquad t = -2$$ Solve for t.

Roller Coasters and Derivatives

Roller coaster rides give you the opportunity to spend a few hair-raising minutes plunging hundreds of feet, accelerating to 80 miles an hour in seven seconds, and enduring vertical loops that turn you upside-down. By finding a function that models your distance above the ground at every moment of the ride and taking its derivative, you can determine when the instantaneous velocity is the greatest. As you experience the glorious agony of the roller coaster, this is your moment of peak terror.

Because we are describing the ball's position for $t \geq 0$, we discard the solution $t = -2$. The ball hits the ground at 5 seconds. Its instantaneous velocity at 5 seconds is found by replacing a with 5 in $s'(a)$.

$$s'(a) = -32a + 48 \qquad \text{This is the ball's instantaneous velocity after } a \text{ seconds.}$$
$$s'(5) = -32 \cdot 5 + 48 = -160 + 48 = -112$$

The instantaneous velocity of the ball when it hits the ground is -112 feet per second. The negative sign indicates that the ball is moving downward at the instant that it strikes the ground.

Check Point 5 A ball is thrown straight up from ground level with an initial velocity of 96 feet per second. The function

$$s(t) = -16t^2 + 96t$$

describes the ball's height above the ground, $s(t)$, in feet, t seconds after it is thrown.

a. What is the instantaneous velocity of the ball after 4 seconds?

b. What is the instantaneous velocity of the ball when it hits the ground?

Exercise Set 11.4

Practice Exercises

In Exercises 1–14,

a. *Find the slope of the tangent line to the graph of f at the given point.*

b. *Find the slope-intercept equation of the tangent line to the graph of f at the given point.*

1. $f(x) = 2x + 3$ at $(1, 5)$

2. $f(x) = 4x + 2$ at $(1, 6)$

3. $f(x) = x^2 + 4$ at $(-1, 5)$

4. $f(x) = x^2 + 7$ at $(-1, 8)$

5. $f(x) = 5x^2$ at $(-2, 20)$

6. $f(x) = 4x^2$ at $(-2, 16)$

7. $f(x) = 2x^2 - x$ at $(2, 6)$

8. $f(x) = 3x^2 + x$ at $(1, 4)$

9. $f(x) = 2x^2 + x - 3$ at $(0, -3)$

10. $f(x) = 2x^2 - x + 5$ at $(0, 5)$

11. $f(x) = \sqrt{x}$ at $(9, 3)$

12. $f(x) = \sqrt{x}$ at $(16, 4)$

13. $f(x) = \frac{1}{x}$ at $(1, 1)$

14. $f(x) = \frac{2}{x}$ at $(1, 2)$

In Exercises 15–28,

a. *Find the derivative of f at x. That is, find $f'(x)$.*

b. *Find the slope of the tangent line to the graph of f at each of the two values of x given to the right of the function.*

15. $f(x) = -3x + 7;\ x = 1, x = 4$

16. $f(x) = -5x + 3;\ x = 1, x = 4$

17. $f(x) = x^2 - 6;\ x = -1, x = 3$

18. $f(x) = x^2 - 8;\ x = -1, x = 3$

19. $f(x) = x^2 - 3x + 5;\ x = \frac{3}{2}, x = 2$

20. $f(x) = x^2 - 4x + 7;\ x = \frac{3}{2}, x = 2$

21. $f(x) = x^3 + 2;\ x = -1, x = 1$

22. $f(x) = x^3 - 2;\ x = -1, x = 1$

23. $f(x) = \sqrt{x};\ x = 1, x = 4$

24. $f(x) = \sqrt{x};\ x = 25, x = 100$

25. $f(x) = \frac{4}{x};\ x = -2, x = 1$

26. $f(x) = \frac{8}{x};\ x = -2, x = 1$

27. $f(x) = 3.2x^2 + 2.1x;\ x = 0, x = 4$

28. $f(x) = 1.3x^2 - 1.4x;\ x = 0, x = 4$

Practice Plus

In Exercises 29–36,

a. *Use transformations of a common graph to obtain the graph of f.*

b. *Find the slope-intercept equation of the tangent line to the graph of f at the point whose x-coordinate is given.*

c. *Use the y-intercept and the slope to graph the tangent line in the same rectangular coordinate system as the graph of f.*

29. $f(x) = (x - 2)^2$; tangent line at 3

30. $f(x) = (x + 2)^2$; tangent line at -1

31. $f(x) = \sqrt{x + 1} - 2$; tangent line at 0

32. $f(x) = \sqrt{x - 1} + 2$; tangent line at 2

33. $f(x) = x^3 + 2$; tangent line at -1

34. $f(x) = x^3 - 2$; tangent line at 1

35. $f(x) = -\frac{1}{x + 3}$; tangent line at -2

36. $f(x) = -\frac{1}{x - 2}$; tangent line at 3

Application Exercises

37. The function $f(x) = x^2$ describes the area of a square, $f(x)$, in square inches, whose sides each measure x inches. If x is changing,

a. Find the average rate of change of the area with respect to x as x changes from 6 inches to 6.1 inches and from 6 inches to 6.01 inches.

b. Find the instantaneous rate of change of the area with respect to x at the moment when $x = 6$ inches.

38. The function $f(x) = x^2$ describes the area of a square, $f(x)$, in square inches, whose sides each measure x inches. If x is changing,

a. Find the average rate of change of the area with respect to x as x changes from 10 inches to 10.1 inches and from 10 inches to 10.01 inches.

b. Find the instantaneous rate of change of the area with respect to x at the moment when $x = 10$ inches.

In Exercises 39–42, express all answers in terms of π.

39. The function $f(x) = \pi x^2$ describes the area of a circle, $f(x)$, in square inches, whose radius measures x inches. If the radius is changing,

a. Find the average rate of change of the area with respect to the radius as the radius changes from 2 inches to 2.1 inches and from 2 inches to 2.01 inches.

b. Find the instantaneous rate of change of the area with respect to the radius when the radius is 2 inches.

40. The function $f(x) = \pi x^2$ describes the area of a circle, $f(x)$, in square inches, whose radius measures x inches. If the radius is changing,

a. Find the average rate of change of the area with respect to the radius as the radius changes from 4 inches to 4.1 inches and from 4 inches to 4.01 inches.

b. Find the instantaneous rate of change of the area with respect to the radius when the radius is 4 inches.

41. The function $f(x) = 4\pi x^2$ describes the surface area, $f(x)$, of a sphere of radius x inches. If the radius is changing, find the instantaneous rate of change of the surface area with respect to the radius when the radius is 6 inches.

42. The function $f(x) = 5\pi x^2$ describes the volume, $f(x)$, of a right circular cylinder of height 5 feet and radius x feet. If the radius is changing, find the instantaneous rate of change of the volume with respect to the radius when the radius is 8 feet.

43. An explosion causes debris to rise vertically with an initial velocity of 64 feet per second. The function

$$s(t) = -16t^2 + 64t$$

describes the height of the debris above the ground, $s(t)$, in feet, t seconds after the explosion.

a. What is the instantaneous velocity of the debris 1 second after the explosion? 3 seconds after the explosion?

b. What is the instantaneous velocity of the debris when it hits the ground?

44. An explosion causes debris to rise vertically with an initial velocity of 72 feet per second. The function

$$s(t) = -16t^2 + 72t$$

describes the height of the debris above the ground, $s(t)$, in feet, t seconds after the explosion.

a. What is the instantaneous velocity of the debris $\frac{1}{2}$ second after the explosion? 4 seconds after the explosion?

b. What is the instantaneous velocity of the debris when it hits the ground?

45. A foul tip of a baseball is hit straight upward from a height of 4 feet with an initial velocity of 96 feet per second. The function

$$s(t) = -16t^2 + 96t + 4$$

describes the ball's height above the ground, $s(t)$, in feet, t seconds after it is hit.

a. What is the instantaneous velocity of the ball 2 seconds after it is hit? 4 seconds after it is hit?

b. The ball reaches its maximum height above the ground when the instantaneous velocity is zero. After how many seconds does the ball reach its maximum height? What is its maximum height?

46. A foul tip of a baseball is hit straight upward from a height of 4 feet with an initial velocity of 64 feet per second. The function

$$s(t) = -16t^2 + 64t + 4$$

describes the ball's height above the ground, $s(t)$, in feet, t seconds after it is hit.

a. What is the instantaneous velocity of the ball 1 second after it is hit? 3 seconds after it is hit?

b. The ball reaches its maximum height above the ground when the instantaneous velocity is zero. After how many seconds does the ball reach its maximum height? What is its maximum height?

Writing in Mathematics

47. Explain how the tangent line to the graph of a function at point P is related to the secant lines between points P and Q on the function's graph.

48. Explain what we mean by the slope of the graph of a function at a point.

49. Explain how to find the slope of $f(x) = x^2$ at $(2, 4)$.

50. Explain how to write an equation of the tangent line to the graph of $f(x) = x^2$ at $(2, 4)$.

51. If you are given $y = f(x)$, the equation of function f, describe how to find $f'(x)$.

52. Explain how to use the derivative to compute the slopes of various tangent lines to the graph of a function.

53. Explain how the instantaneous rate of change of a function at a point is related to its average rates of change.

54. If a function expresses an object's position in terms of time, how do you find the instantaneous velocity of the object at any time during its motion?

55. Use the concept of an interval of time to describe how calculus views a particular instant of time.

56. You are about to take a great picture of fog rolling into San Francisco from the middle of the Golden Gate Bridge, 400 feet above the water. Whoops! You accidently lean too far over the safety rail and drop your camera. Your friend quips, "Well at least you know calculus; you can figure out the velocity with which the camera is going to hit the water." If the camera's height, $s(t)$, in feet, over the water after t seconds is $s(t) = 400 - 16t^2$, describe how to determine the camera's velocity at the instant of its demise.

57. A calculus professor introduced the derivative by saying that it could be summed up in one word: *slope*. Explain what this means.

58. For an unusual introduction to calculus by a poetic, quirky, and funny writer who loves the subject, read *A Tour of the Calculus* by David Berlinski (Vintage Books, 1995). Write a report describing two new things that you learned from the book about algebra, trigonometry, limits, or derivatives.

Technology Exercises

59. Use the DRAW TANGENT feature of a graphing utility to graph the functions and tangent lines for any five exercises from Exercises 1–14. Use the equation that is displayed on the screen to verify the slope-intercept equation of the tangent line that you found in each exercise.

60. Without using the DRAW TANGENT feature of a graphing utility, graph the function and the tangent line whose equation you found for any five exercises from Exercises 1–14. Does the line appear to be tangent to the graph of f at the point on f that is given in the exercise?

61. Use the feature on a graphing utility that gives the derivative of a function evaluated at any number to verify part (b) for any five of your answers in Exercises 15–28.

In Exercises 62–65, find, or approximate to two decimal places, the derivative of each function at the given number using a graphing utility.

62. $f(x) = x^4 - x^3 + x^2 - x + 1$ at 1

63. $f(x) = \dfrac{x}{x-3}$ at 6

64. $f(x) = x^2 \cos x$ at $\dfrac{\pi}{4}$

65. $f(x) = e^x \sin x$ at 2

Critical Thinking Exercises

Make Sense? *In Exercises 66–69, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

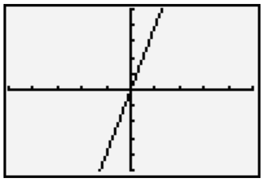
66. Because I have two points to work with, I use the formula for slope, $\dfrac{y_2 - y_1}{x_2 - x_1}$, to find the slope of the tangent line to the graph of a function $y = f(x)$ at $(a, f(a))$.

67. I can find the slope of the tangent line to the graph of $f(x)$ at $(3, f(3))$ using $\lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$ or finding $f'(x)$ and then replacing x with 3.

68. I obtained $f'(x)$ by finding $\lim\limits_{h\to 0}[f(x+h) - f(x)]$ and $\lim\limits_{h\to 0} h$, and then using the quotient rule for limits.

69. If $f(x) = \pi x^2$ describes the area of a circle, $f(x)$, with radius x, $f'(5) > f'(2)$ because the area increases more rapidly as the radius increases.

In Exercises 70–75, graphs of functions are shown in $[-5, 5, 1]$ *by* $[-5, 5, 1]$ *viewing rectangles. Match each function with the graph of its derivative. Graphs of derivatives are labeled (a)–(f) and are shown in* $[-5, 5, 1]$ *by* $[-5, 5, 1]$ *viewing rectangles.*

70.


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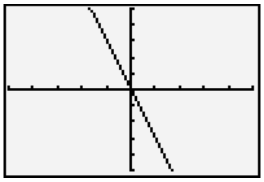

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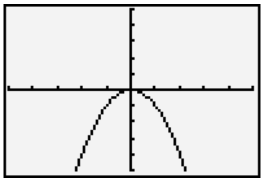

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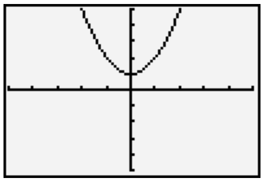

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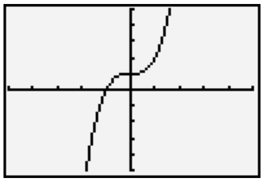

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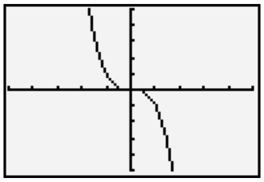

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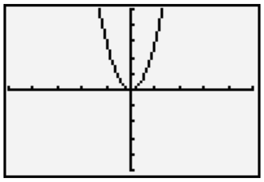

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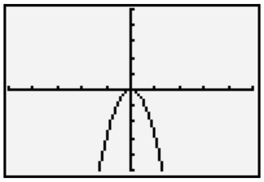

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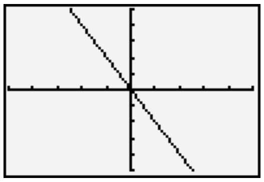

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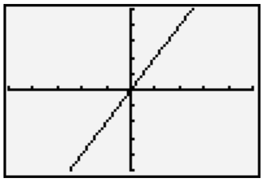

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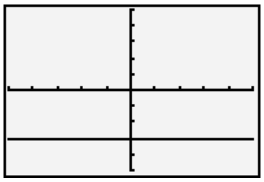

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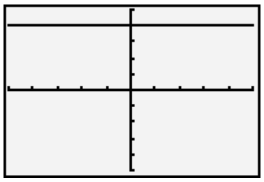

76. A ball is thrown straight up from a rooftop 96 feet high with an initial velocity of 80 feet per second. The function

$$s(t) = -16t^2 + 80t + 96$$

describes the ball's height above the ground, $s(t)$, in feet, t seconds after it is thrown. The ball misses the rooftop on its way down and eventually strikes the ground. What is its instantaneous velocity as it passes the rooftop on the way down?

77. Show that the rate of change of the area of a circle with respect to its radius is equal to the circumference of the circle.

78. Show that the x-coordinate of the vertex of the parabola whose equation is $y = ax^2 + bx + c$ occurs when the derivative of the function is zero.

79. For any positive integer n, prove that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.