92. Graph $f(x) = 2^x$ and its inverse function in the same rectangular coordinate system.

93. The *hyperbolic cosine* and *hyperbolic sine* functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

a. Show that $\cosh x$ is an even function.
b. Show that $\sinh x$ is an odd function.
c. Prove that $(\cosh x)^2 - (\sinh x)^2 = 1$.

Preview Exercises

Exercises 94–96 will help you prepare for the material covered in the next section.

94. What problem do you encounter when using the switch-and-solve strategy to find the inverse of $f(x) = 2^x$? (The switch-and-solve strategy is described in the box on page 235.)

95. 25 to what power gives 5? $(25^? = 5)$

96. Solve: $(x - 3)^2 > 0$.

Section 3.2 Logarithmic Functions

Objectives

1. Change from logarithmic to exponential form.
2. Change from exponential to logarithmic form.
3. Evaluate logarithms.
4. Use basic logarithmic properties.
5. Graph logarithmic functions.
6. Find the domain of a logarithmic function.
7. Use common logarithms.
8. Use natural logarithms.

The earthquake that ripped through northern California on October 17, 1989 measured 7.1 on the Richter scale, killed more than 60 people, and injured more than 2400. Shown here is San Francisco's Marina district, where shock waves tossed houses off their foundations and into the street.

A higher measure on the Richter scale is more devastating than it seems because for each increase in one unit on the scale, there is a tenfold increase in the intensity of an earthquake. In this section, our focus is on the inverse of the exponential function, called the logarithmic function. The logarithmic function will help you to understand diverse phenomena, including earthquake intensity, human memory, and the pace of life in large cities.

Study Tip

The discussion that follows is based on our work with inverse functions in Section 1.8. Here is a summary of what you should already know about functions and their inverses.

1. Only one-to-one functions have inverses that are functions. A function, f, has an inverse function, f^{-1}, if there is no horizontal line that intersects the graph of f at more than one point.
2. If a function is one-to-one, its inverse function can be found by interchanging x and y in the function's equation and solving for y.
3. If $f(a) = b$, then $f^{-1}(b) = a$. The domain of f is the range of f^{-1}. The range of f is the domain of f^{-1}.
4. $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
5. The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

The Definition of Logarithmic Functions

No horizontal line can be drawn that intersects the graph of an exponential function at more than one point. This means that the exponential function is one-to-one and has an inverse. Let's use our switch-and-solve strategy from Section 1.8 to find the inverse.

All exponential functions have inverse functions.

$$f(x) = b^x$$

Step 1 Replace $f(x)$ with y: $y = b^x$.
Step 2 Interchange x and y: $x = b^y$.
Step 3 Solve for y: ?

The question mark indicates that we do not have a method for solving $b^y = x$ for y. To isolate the exponent y, a new notation, called *logarithmic notation*, is needed. This notation gives us a way to name the inverse of $f(x) = b^x$. **The inverse function of the exponential function with base *b* is called the *logarithmic function with base b*.**

Definition of the Logarithmic Function

For $x > 0$ and $b > 0, b \neq 1$,

$$y = \log_b x \text{ is equivalent to } b^y = x.$$

The function $f(x) = \log_b x$ is the **logarithmic function with base *b***.

The equations

$$y = \log_b x \quad \text{and} \quad b^y = x$$

are different ways of expressing the same thing. The first equation is in **logarithmic form** and the second equivalent equation is in **exponential form**.

Notice that a **logarithm, *y*, is an exponent**. You should learn the location of the base and exponent in each form.

Study Tip

To change from logarithmic form to the more familiar exponential form, use this pattern:

$$y = \log_b x \quad \text{means} \quad b^y = x.$$

Location of Base and Exponent in Exponential and Logarithmic Forms

Logarithmic Form: $y = \log_b x$ (Exponent: y; Base: b) Exponential Form: $b^y = x$ (Exponent: y; Base: b)

1 Change from logarithmic to exponential form.

EXAMPLE 1 Changing from Logarithmic to Exponential Form

Write each equation in its equivalent exponential form:

a. $2 = \log_5 x$ **b.** $3 = \log_b 64$ **c.** $\log_3 7 = y$.

Solution We use the fact that $y = \log_b x$ means $b^y = x$.

a. $2 = \log_5 x$ means $5^2 = x$. (Logarithms are exponents.)

b. $3 = \log_b 64$ means $b^3 = 64$. (Logarithms are exponents.)

c. $\log_3 7 = y$ or $y = \log_3 7$ means $3^y = 7$.

Check Point 1 Write each equation in its equivalent exponential form:

a. $3 = \log_7 x$ **b.** $2 = \log_b 25$ **c.** $\log_4 26 = y$.

2 Change from exponential to logarithmic form.

EXAMPLE 2 Changing from Exponential to Logarithmic Form

Write each equation in its equivalent logarithmic form:

a. $12^2 = x$ **b.** $b^3 = 8$ **c.** $e^y = 9$.

Solution We use the fact that $b^y = x$ means $y = \log_b x$.

a. $12^2 = x$ means $2 = \log_{12} x$. (Exponents are logarithms.)

b. $b^3 = 8$ means $3 = \log_b 8$. (Exponents are logarithms.)

c. $e^y = 9$ means $y = \log_e 9$.

Check Point 2 Write each equation in its equivalent logarithmic form:

a. $2^5 = x$ b. $b^3 = 27$ c. $e^y = 33$.

3 Evaluate logarithms.

Remembering that logarithms are exponents makes it possible to evaluate some logarithms by inspection. The logarithm of x with base b, $\log_b x$, is the exponent to which b must be raised to get x. For example, suppose we want to evaluate $\log_2 32$. We ask, 2 to what power gives 32? Because $2^5 = 32$, $\log_2 32 = 5$.

EXAMPLE 3 Evaluating Logarithms

Evaluate:

a. $\log_2 16$ b. $\log_7 \frac{1}{49}$ c. $\log_{25} 5$ d. $\log_2 \sqrt[5]{2}$.

Solution

Logarithmic Expression	Question Needed for Evaluation	Logarithmic Expression Evaluated
a. $\log_2 16$	2 to what power gives 16?	$\log_2 16 = 4$ because $2^4 = 16$.
b. $\log_7 \frac{1}{49}$	7 to what power gives $\frac{1}{49}$?	$\log_7 \frac{1}{49} = -2$ because $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$.
c. $\log_{25} 5$	25 to what power gives 5?	$\log_{25} 5 = \frac{1}{2}$ because $25^{\frac{1}{2}} = \sqrt{25} = 5$.
d. $\log_2 \sqrt[5]{2}$	2 to what power gives $\sqrt[5]{2}$, or $2^{\frac{1}{5}}$?	$\log_2 \sqrt[5]{2} = \frac{1}{5}$ because $2^{\frac{1}{5}} = \sqrt[5]{2}$.

Check Point 3 Evaluate:

a. $\log_{10} 100$ b. $\log_5 \frac{1}{125}$ c. $\log_{36} 6$ d. $\log_3 \sqrt[7]{3}$.

4 Use basic logarithmic properties.

Basic Logarithmic Properties

Because logarithms are exponents, they have properties that can be verified using properties of exponents.

Basic Logarithmic Properties Involving One

1. $\log_b b = 1$ because 1 is the exponent to which b must be raised to obtain b. $(b^1 = b)$
2. $\log_b 1 = 0$ because 0 is the exponent to which b must be raised to obtain 1. $(b^0 = 1)$

EXAMPLE 4 Using Properties of Logarithms

Evaluate:

a. $\log_7 7$ b. $\log_5 1$.

Solution

a. Because $\log_b b = 1$, we conclude $\log_7 7 = 1$. This means that $7^1 = 7$.

b. Because $\log_b 1 = 0$, we conclude $\log_5 1 = 0$. This means that $5^0 = 1$.

Check Point 4 Evaluate:

a. $\log_9 9$ **b.** $\log_8 1$.

Now that we are familiar with logarithmic notation, let's resume and finish the switch-and-solve strategy for finding the inverse of $f(x) = b^x$.

Step 1 **Replace $f(x)$ with y:** $y = b^x$.

Step 2 **Interchange x and y:** $x = b^y$.

Step 3 **Solve for y:** $y = \log_b x$.

Step 4 **Replace y with $f^{-1}(x)$:** $f^{-1}(x) = \log_b x$.

The completed switch-and-solve strategy illustrates that if $f(x) = b^x$, then $f^{-1}(x) = \log_b x$. The inverse of an exponential function is the logarithmic function with the same base.

In Section 1.8, we saw how inverse functions "undo" one another. In particular,

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Applying these relationships to exponential and logarithmic functions, we obtain the following **inverse properties of logarithms**:

Inverse Properties of Logarithms

For $b > 0$ and $b \neq 1$,

$\log_b b^x = x$ The logarithm with base b of b raised to a power equals that power.

$b^{\log_b x} = x.$ b raised to the logarithm with base b of a number equals that number.

EXAMPLE 5 Using Inverse Properties of Logarithms

Evaluate:

a. $\log_4 4^5$ **b.** $6^{\log_6 9}$.

Solution

a. Because $\log_b b^x = x$, we conclude $\log_4 4^5 = 5$.

b. Because $b^{\log_b x} = x$, we conclude $6^{\log_6 9} = 9$.

Check Point 5 Evaluate:

a. $\log_7 7^8$ **b.** $3^{\log_3 17}$.

5 Graph logarithmic functions.

Graphs of Logarithmic Functions

How do we graph logarithmic functions? We use the fact that a logarithmic function is the inverse of an exponential function. This means that the logarithmic function reverses the coordinates of the exponential function. It also means that the graph of the logarithmic function is a reflection of the graph of the exponential function about the line $y = x$.

EXAMPLE 6 Graphs of Exponential and Logarithmic Functions

Graph $f(x) = 2^x$ and $g(x) = \log_2 x$ in the same rectangular coordinate system.

Solution We first set up a table of coordinates for $f(x) = 2^x$. Reversing these coordinates gives the coordinates for the inverse function $g(x) = \log_2 x$.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Reverse coordinates.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x) = \log_2 x$	-2	-1	0	1	2	3

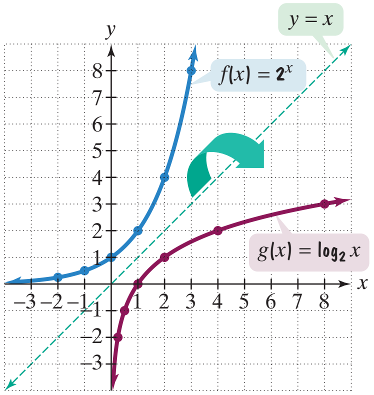


Figure 3.7 The graphs of $f(x) = 2^x$ and its inverse function

We now plot the ordered pairs from each table, connecting them with smooth curves. **Figure 3.7** shows the graphs of $f(x) = 2^x$ and its inverse function $g(x) = \log_2 x$. The graph of the inverse can also be drawn by reflecting the graph of $f(x) = 2^x$ about the line $y = x$.

Study Tip

You can obtain a partial table of coordinates for $g(x) = \log_2 x$ without having to obtain and reverse coordinates for $f(x) = 2^x$. Because $g(x) = \log_2 x$ means $2^{g(x)} = x$, we begin with values for $g(x)$ and compute corresponding values for x:

Use $x = 2^{g(x)}$ to compute x. For example, if $g(x) = -2$, $x = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$.

Start with values for $g(x)$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x) = \log_2 x$	-2	-1	0	1	2	3

Check Point 6 Graph $f(x) = 3^x$ and $g(x) = \log_3 x$ in the same rectangular coordinate system.

Figure 3.8 illustrates the relationship between the graph of an exponential function, shown in blue, and its inverse, a logarithmic function, shown in red, for bases greater than 1 and for bases between 0 and 1. Also shown and labeled are the exponential function's horizontal asymptote ($y = 0$) and the logarithmic function's vertical asymptote ($x = 0$).

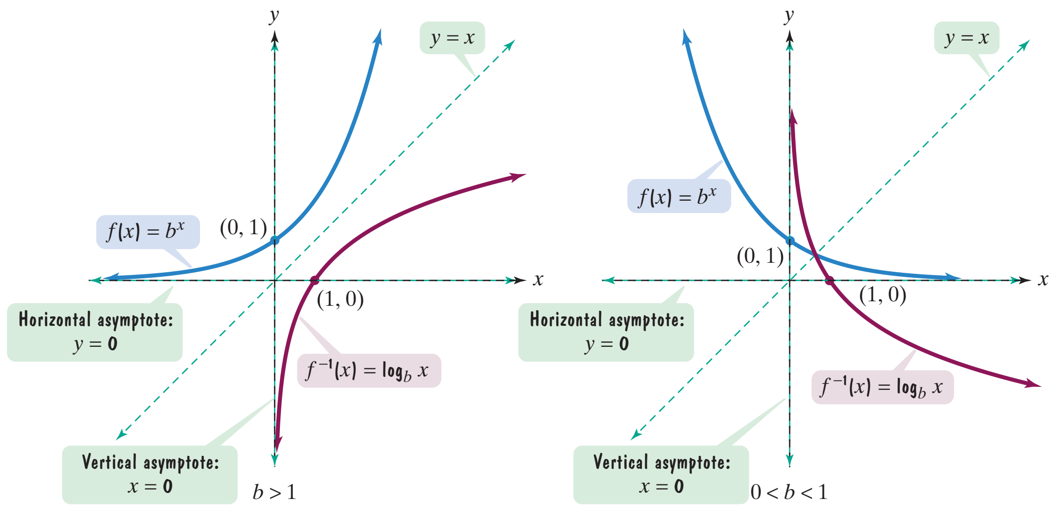


Figure 3.8 Graphs of exponential and logarithmic functions

The red graphs in **Figure 3.8** illustrate the following general characteristics of logarithmic functions:

Characteristics of Logarithmic Functions of the Form $f(x) = \log_b x$

1. The domain of $f(x) = \log_b x$ consists of all positive real numbers: $(0, \infty)$. The range of $f(x) = \log_b x$ consists of all real numbers: $(-\infty, \infty)$.
2. The graphs of all logarithmic functions of the form $f(x) = \log_b x$ pass through the point $(1, 0)$ because $f(1) = \log_b 1 = 0$. The x-intercept is 1. There is no y-intercept.
3. If $b > 1$, $f(x) = \log_b x$ has a graph that goes up to the right and is an increasing function.
4. If $0 < b < 1$, $f(x) = \log_b x$ has a graph that goes down to the right and is a decreasing function.
5. The graph of $f(x) = \log_b x$ approaches, but does not touch, the y-axis. The y-axis, or $x = 0$, is a vertical asymptote. As $x \to 0^+$, $\log_b x \to -\infty$ or ∞.

The graphs of logarithmic functions can be translated vertically or horizontally, reflected, stretched, or shrunk. These transformations are summarized in **Table 3.4**.

Table 3.4 Transformations Involving Logarithmic Functions

In each case, c represents a positive real number.

Transformation	Equation	Description
Vertical translation	$g(x) = \log_b x + c$ $g(x) = \log_b x - c$	• Shifts the graph of $f(x) = \log_b x$ upward c units. • Shifts the graph of $f(x) = \log_b x$ downward c units.
Horizontal translation	$g(x) = \log_b(x + c)$ $g(x) = \log_b(x - c)$	• Shifts the graph of $f(x) = \log_b x$ to the left c units. Vertical asymptote: $x = -c$ • Shifts the graph of $f(x) = \log_b x$ to the right c units. Vertical asymptote: $x = c$
Reflection	$g(x) = -\log_b x$ $g(x) = \log_b(-x)$	• Reflects the graph of $f(x) = \log_b x$ about the x-axis. • Reflects the graph of $f(x) = \log_b x$ about the y-axis.
Vertical stretching or shrinking	$g(x) = c \log_b x$	• Vertically stretches the graph of $f(x) = \log_b x$ if $c > 1$. • Vertically shrinks the graph of $f(x) = \log_b x$ if $0 < c < 1$.
Horizontal stretching or shrinking	$g(x) = \log_b(cx)$	• Horizontally shrinks the graph of $f(x) = \log_b x$ if $c > 1$. • Horizontally stretches the graph of $f(x) = \log_b x$ if $0 < c < 1$.

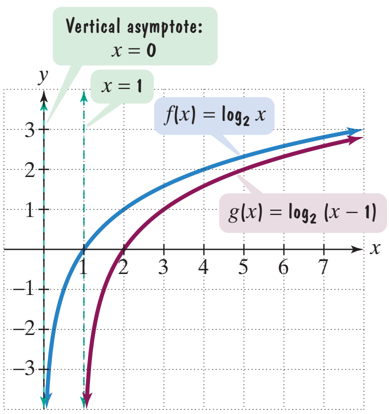


Figure 3.9 Shifting $f(x) = \log_2 x$ one unit to the right

For example, **Figure 3.9** illustrates that the graph of $g(x) = \log_2(x - 1)$ is the graph of $f(x) = \log_2 x$ shifted one unit to the right. If a logarithmic function is translated to the left or to the right, both the x-intercept and the vertical asymptote are shifted by the amount of the horizontal shift. In **Figure 3.9**, the x-intercept of f is 1. Because g is shifted one unit to the right, its x-intercept is 2. Also observe that the vertical asymptote for f, the y-axis, or $x = 0$, is shifted one unit to the right for the vertical asymptote for g. Thus, $x = 1$ is the vertical asymptote for g.

Here are some other examples of transformations of graphs of logarithmic functions:

- The graph of $g(x) = 3 + \log_4 x$ is the graph of $f(x) = \log_4 x$ shifted up three units, shown in **Figure 3.10**.
- The graph of $h(x) = -\log_2 x$ is the graph of $f(x) = \log_2 x$ reflected about the x-axis, shown in **Figure 3.11**.
- The graph of $r(x) = \log_2(-x)$ is the graph of $f(x) = \log_2 x$ reflected about the y-axis, shown in **Figure 3.12**.

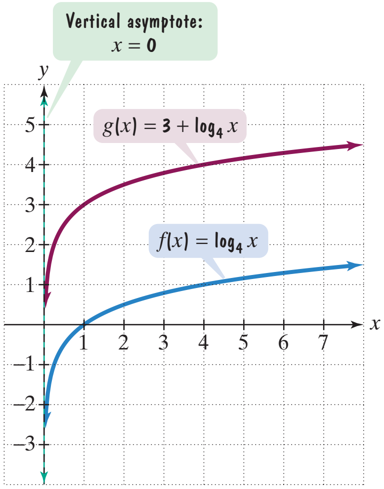


Figure 3.10 Shifting vertically up three units

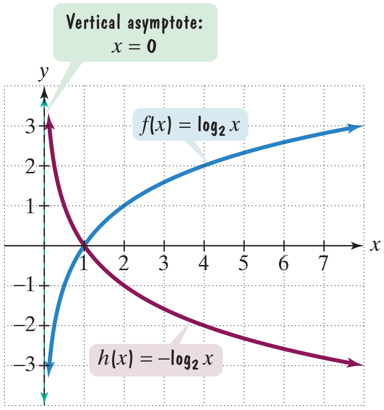


Figure 3.11 Reflection about the x-axis

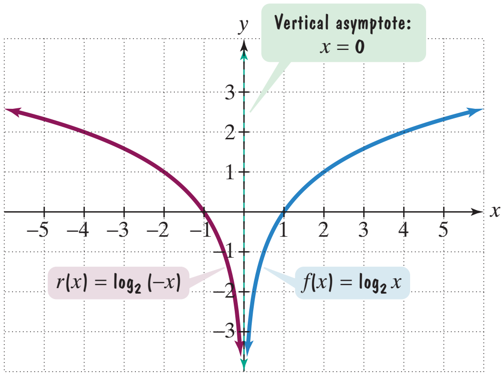


Figure 3.12 Reflection about the y-axis

6 Find the domain of a logarithmic function.

The Domain of a Logarithmic Function

In Section 3.1, we learned that the domain of an exponential function of the form $f(x) = b^x$ includes all real numbers and its range is the set of positive real numbers. Because the logarithmic function reverses the domain and the range of the exponential function, the **domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers**. Thus, $\log_2 8$ is defined because the value of x in the logarithmic expression, 8, is greater than zero and therefore is included in the domain of the logarithmic function $f(x) = \log_2 x$. However, $\log_2 0$ and $\log_2(-8)$ are not defined because 0 and -8 are not positive real numbers and therefore are excluded from the domain of the logarithmic function $f(x) = \log_2 x$. In general, **the domain of $f(x) = \log_b g(x)$ consists of all x for which $g(x) > 0$.**

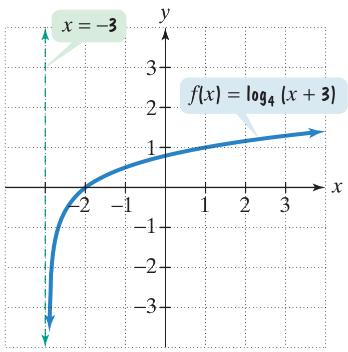


Figure 3.13 The domain of $f(x) = \log_4(x + 3)$ is $(-3, \infty)$.

EXAMPLE 7 Finding the Domain of a Logarithmic Function

Find the domain of $f(x) = \log_4(x + 3)$.

Solution The domain of f consists of all x for which $x + 3 > 0$. Solving this inequality for x, we obtain $x > -3$. Thus, the domain of f is $(-3, \infty)$. This is illustrated in **Figure 3.13**. The vertical asymptote is $x = -3$ and all points on the graph of f have x-coordinates that are greater than -3.

Check Point 7 Find the domain of $f(x) = \log_4(x - 5)$.

7 Use common logarithms.

Common Logarithms

The logarithmic function with base 10 is called the **common logarithmic function**. The function $f(x) = \log_{10} x$ is usually expressed as $f(x) = \log x$. A calculator with a LOG key can be used to evaluate common logarithms. Here are some examples:

Logarithm	**Most Scientific Calculator Keystrokes**	**Most Graphing Calculator Keystrokes**	**Display (or Approximate Display)**
$\log 1000$	1000 [LOG]	[LOG] 1000 [ENTER]	3
$\log \frac{5}{2}$	[(] 5 [÷] 2 [)] [LOG]	[LOG] [(] 5 [÷] 2 [)] [ENTER]	0.39794
$\frac{\log 5}{\log 2}$	5 [LOG] [÷] 2 [LOG] [=]	[LOG] 5 [÷] [LOG] 2 [ENTER]	2.32193
$\log(-3)$	3 [+/−] [LOG]	[LOG] [(−)] 3 [ENTER]	[ERROR]

Some graphing calculators display an open parenthesis when the [LOG] key is pressed. In this case, remember to close the set of parentheses after entering the function's domain value: [LOG] 5 [)] [÷] [LOG] 2 [)] [ENTER].

The error message or [NONREAL ANS] given by many calculators for $\log(-3)$ is a reminder that the domain of the common logarithmic function, $f(x) = \log x$, is the set of positive real numbers. In general, the domain of $f(x) = \log g(x)$ consists of all x for which $g(x) > 0$.

Many real-life phenomena start with rapid growth and then the growth begins to level off. This type of behavior can be modeled by logarithmic functions.

EXAMPLE 8 Modeling Height of Children

The percentage of adult height attained by a boy who is x years old can be modeled by

$$f(x) = 29 + 48.8 \log(x + 1),$$

where x represents the boy's age and $f(x)$ represents the percentage of his adult height. Approximately what percentage of his adult height has a boy attained at age eight?

Solution We substitute the boy's age, 8, for x and evaluate the function.

$$f(x) = 29 + 48.8 \log(x + 1)$$ This is the given function.

$$f(8) = 29 + 48.8 \log(8 + 1)$$ Substitute 8 for x.

$$= 29 + 48.8 \log 9$$ Graphing calculator keystrokes: 29 [+] 48.8 [LOG] 9 [ENTER]

$$\approx 76$$

Thus, an 8-year-old boy has attained approximately 76% of his adult height.

Check Point 8 Use the function in Example 8 to answer this question: Approximately what percentage of his adult height has a boy attained at age ten?

The basic properties of logarithms that were listed earlier in this section can be applied to common logarithms.

Properties of Common Logarithms

General Properties	**Common Logarithms**
1. $\log_b 1 = 0$	**1.** $\log 1 = 0$
2. $\log_b b = 1$	**2.** $\log 10 = 1$
3. $\log_b b^x = x$	**3.** $\log 10^x = x$
4. $b^{\log_b x} = x$	**4.** $10^{\log x} = x$

Inverse properties (3 and 4)

The property $\log 10^x = x$ can be used to evaluate common logarithms involving powers of 10. For example,

$$\log 100 = \log 10^2 = 2, \quad \log 1000 = \log 10^3 = 3, \quad \text{and} \quad \log 10^{7.1} = 7.1.$$

EXAMPLE 9 Earthquake Intensity

The magnitude, R, on the Richter scale of an earthquake of intensity I is given by

$$R = \log\frac{I}{I_0},$$

where I_0 is the intensity of a barely felt zero-level earthquake. The earthquake that destroyed San Francisco in 1906 was $10^{8.3}$ times as intense as a zero-level earthquake. What was its magnitude on the Richter scale?

Solution Because the earthquake was $10^{8.3}$ times as intense as a zero-level earthquake, the intensity, I, is $10^{8.3}\, I_0$.

$R = \log\frac{I}{I_0}$ — This is the formula for magnitude on the Richter scale.

$R = \log\frac{10^{8.3}I_0}{I_0}$ — Substitute $10^{8.3}I_0$ for I.

$= \log 10^{8.3}$ — Simplify.

$= 8.3$ — Use the property $\log 10^x = x$.

San Francisco's 1906 earthquake registered 8.3 on the Richter scale.

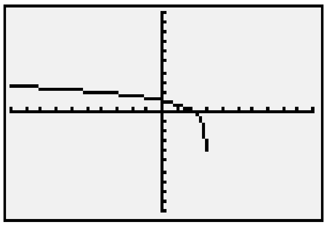
Check Point 9 Use the formula in Example 9 to solve this problem. If an earthquake is 10,000 times as intense as a zero-level quake ($I = 10{,}000I_0$), what is its magnitude on the Richter scale?

8 Use natural logarithms.

Natural Logarithms

The logarithmic function with base e is called the **natural logarithmic function**. The function $f(x) = \log_e x$ is usually expressed as $f(x) = \ln x$, read "el en of x." A calculator with an LN key can be used to evaluate natural logarithms. Keystrokes are identical to those shown for common logarithmic evaluations on page 407.

Like the domain of all logarithmic functions, the domain of the natural logarithmic function $f(x) = \ln x$ is the set of all positive real numbers. Thus, the domain of $f(x) = \ln g(x)$ consists of all x for which $g(x) > 0$.



[−10, 10, 1] by [−10, 10, 1]

Figure 3.14 The domain of $f(x) = \ln(3 - x)$ is $(-\infty, 3)$.

EXAMPLE 10 Finding Domains of Natural Logarithmic Functions

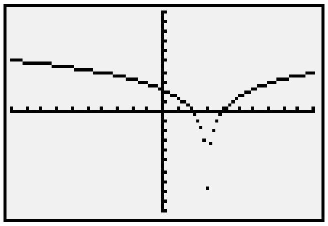
Find the domain of each function:

a. $f(x) = \ln(3 - x)$ **b.** $h(x) = \ln(x - 3)^2$.

Solution

a. The domain of f consists of all x for which $3 - x > 0$. Solving this inequality for x, we obtain $x < 3$. Thus, the domain of f is $\{x | x < 3\}$ or $(-\infty, 3)$. This is verified by the graph in **Figure 3.14**.

b. The domain of h consists of all x for which $(x - 3)^2 > 0$. It follows that the domain of h is all real numbers except 3. Thus, the domain of h is $\{x | x \neq 3\}$ or $(-\infty, 3) \cup (3, \infty)$. This is shown by the graph in **Figure 3.15**. To make it more obvious that 3 is excluded from the domain, we used a DOT format.



[−10, 10, 1] by [−10, 10, 1]

Figure 3.15 3 is excluded from the domain of $h(x) = \ln(x - 3)^2$.

Check Point 10 Find the domain of each function:

a. $f(x) = \ln(4 - x)$ **b.** $h(x) = \ln x^2$.

The basic properties of logarithms that were listed earlier in this section can be applied to natural logarithms.

Properties of Natural Logarithms

General Properties		Natural Logarithms
1. $\log_b 1 = 0$		**1.** $\ln 1 = 0$
2. $\log_b b = 1$		**2.** $\ln e = 1$
3. $\log_b b^x = x$	Inverse properties	**3.** $\ln e^x = x$
4. $b^{\log_b x} = x$		**4.** $e^{\ln x} = x$

Examine the inverse properties, $\ln e^x = x$ and $e^{\ln x} = x$. Can you see how ln and e "undo" one another? For example,

$$\ln e^2 = 2, \quad \ln e^{7x^2} = 7x^2, \quad e^{\ln 2} = 2, \quad \text{and} \quad e^{\ln 7x^2} = 7x^2.$$

EXAMPLE 11 Dangerous Heat: Temperature in an Enclosed Vehicle

When the outside air temperature is anywhere from 72° to 96° Fahrenheit, the temperature in an enclosed vehicle climbs by 43° in the first hour. The bar graph in **Figure 3.16** shows the temperature increase throughout the hour. The function

$$f(x) = 13.4 \ln x - 11.6$$

models the temperature increase, $f(x)$, in degrees Fahrenheit, after x minutes. Use the function to find the temperature increase, to the nearest degree, after 50 minutes. How well does the function model the actual increase shown in **Figure 3.16**?

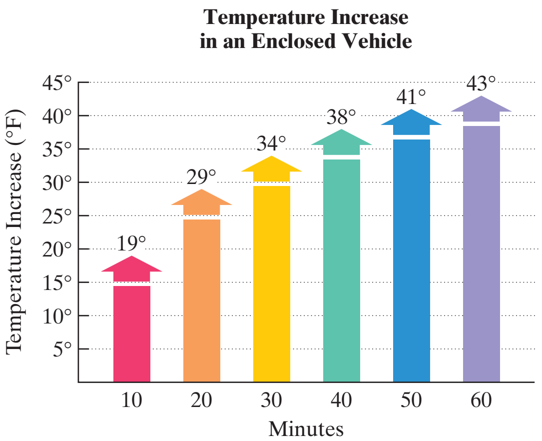


Figure 3.16
Source: Professor Jan Null, San Francisco State University

Solution We find the temperature increase after 50 minutes by substituting 50 for x and evaluating the function at 50.

$$f(x) = 13.4 \ln x - 11.6$$ This is the given function.

$$f(50) = 13.4 \ln 50 - 11.6$$ Substitute 50 for x.

$$\approx 41$$ Graphing calculator keystrokes: 13.4 [ln] 50 [−] 11.6 [ENTER]. On some calculators, a parenthesis is needed after 50.

According to the function, the temperature will increase by approximately 41° after 50 minutes. Because the increase shown in **Figure 3.16** is 41°, the function models the actual increase extremely well.

Check Point 11 Use the function in Example 11 to find the temperature increase, to the nearest degree, after 30 minutes. How well does the function model the actual increase shown in **Figure 3.16**?

The Curious Number e

You will learn more about each curiosity mentioned below when you take calculus.

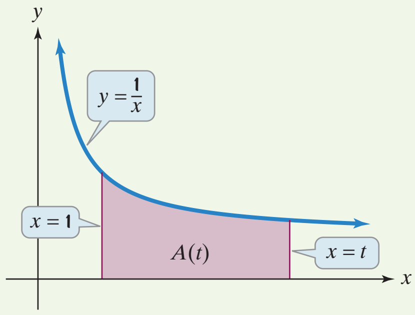


Figure 3.17

- The number e was named by the Swiss mathematician Leonhard Euler (1707–1783), who proved that it is the limit as $n \to \infty$ of $\left(1 + \frac{1}{n}\right)^n$.
- e features in Euler's remarkable relationship $e^{i\pi} = -1$, in which $i = \sqrt{-1}$.
- The first few decimal places of e are fairly easy to remember: $e = 2.7\ 1828\ 1828\ 45\ 90\ 45\ldots.$
- The best rational approximation of e using numbers less than 1000 is also easy to remember: $e \approx \frac{878}{323} \approx 2.71826\ldots.$
- Isaac Newton (1642–1727), one of the cofounders of calculus, showed that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, from which we obtain $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$, an infinite sum suitable for calculation because its terms decrease so rapidly. (*Note*: $n!$ (n factorial) is the product of all the consecutive integers from n down to 1: $n! = n(n-1)(n-2)(n-3)\cdot \cdots \cdot 3 \cdot 2 \cdot 1$.)
- The area of the region bounded by $y = \frac{1}{x}$, the x-axis, $x = 1$ and $x = t$ (shaded in **Figure 3.17**) is a function of t, designated by $A(t)$. Grégoire de Saint-Vincent, a Belgian Jesuit (1584–1667), spent his entire professional life attempting to find a formula for $A(t)$. With his student, he showed that $A(t) = \ln t$, becoming one of the first mathematicians to make use of the logarithmic function for something other than a computational device.

Exercise Set 3.2

Practice Exercises

In Exercises 1–8, write each equation in its equivalent exponential form.

1. $4 = \log_2 16$ **2.** $6 = \log_2 64$

3. $2 = \log_3 x$ **4.** $2 = \log_9 x$

5. $5 = \log_b 32$ **6.** $3 = \log_b 27$

7. $\log_6 216 = y$ **8.** $\log_5 125 = y$

In Exercises 9–20, write each equation in its equivalent logarithmic form.

9. $2^3 = 8$ **10.** $5^4 = 625$ **11.** $2^{-4} = \frac{1}{16}$ **12.** $5^{-3} = \frac{1}{125}$

13. $\sqrt[3]{8} = 2$ **14.** $\sqrt[3]{64} = 4$ **15.** $13^2 = x$ **16.** $15^2 = x$

17. $b^3 = 1000$ **18.** $b^3 = 343$ **19.** $7^y = 200$ **20.** $8^y = 300$

In Exercises 21–42, evaluate each expression without using a calculator.

21. $\log_4 16$ **22.** $\log_7 49$ **23.** $\log_2 64$ **24.** $\log_3 27$

25. $\log_5 \frac{1}{5}$ **26.** $\log_6 \frac{1}{6}$ **27.** $\log_2 \frac{1}{8}$ **28.** $\log_3 \frac{1}{9}$

29. $\log_7 \sqrt{7}$ **30.** $\log_6 \sqrt{6}$ **31.** $\log_2 \frac{1}{\sqrt{2}}$ **32.** $\log_3 \frac{1}{\sqrt{3}}$

33. $\log_{64} 8$ **34.** $\log_{81} 9$ **35.** $\log_5 5$ **36.** $\log_{11} 11$

37. $\log_4 1$ **38.** $\log_6 1$ **39.** $\log_5 5^7$ **40.** $\log_4 4^6$

41. $8^{\log_8 19}$ **42.** $7^{\log_7 23}$

43. Graph $f(x) = 4^x$ and $g(x) = \log_4 x$ in the same rectangular coordinate system.

44. Graph $f(x) = 5^x$ and $g(x) = \log_5 x$ in the same rectangular coordinate system.

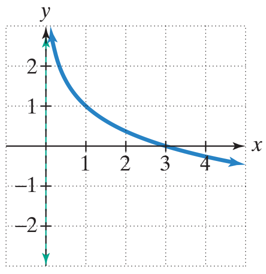
45. Graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \log_{\frac{1}{2}} x$ in the same rectangular coordinate system.

46. Graph $f(x) = \left(\frac{1}{4}\right)^x$ and $g(x) = \log_{\frac{1}{4}} x$ in the same rectangular coordinate system.

In Exercises 47–52, the graph of a logarithmic function is given. Select the function for each graph from the following options:

$$f(x) = \log_3 x,\ g(x) = \log_3(x-1),\ h(x) = \log_3 x - 1,$$
$$F(x) = -\log_3 x,\ G(x) = \log_3(-x),\ H(x) = 1 - \log_3 x.$$

47.



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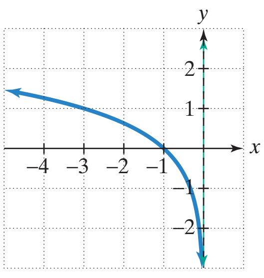

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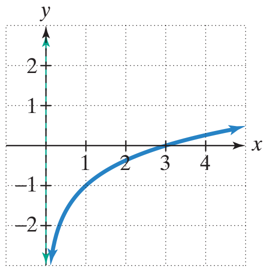
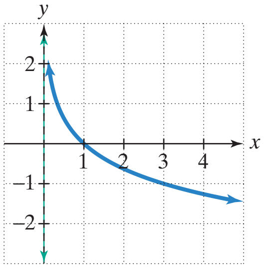
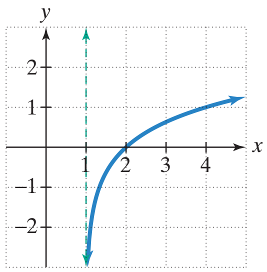


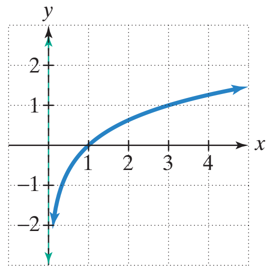
50.



51.



52.



In Exercises 53–58, begin by graphing $f(x) = \log_2 x$. Then use transformations of this graph to graph the given function. What is the vertical asymptote? Use the graphs to determine each function's domain and range.

53. $g(x) = \log_2(x + 1)$

54. $g(x) = \log_2(x + 2)$

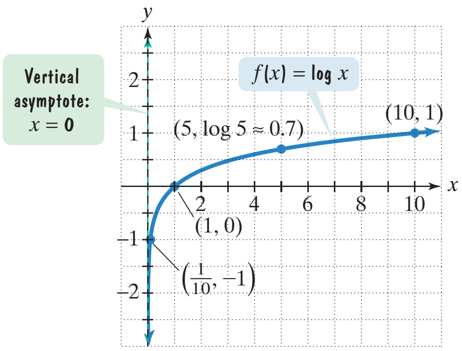
55. $h(x) = 1 + \log_2 x$

56. $h(x) = 2 + \log_2 x$

57. $g(x) = \frac{1}{2}\log_2 x$

58. $g(x) = -2 \log_2 x$

The figure shows the graph of $f(x) = \log x$. In Exercises 59–64, use transformations of this graph to graph each function. Graph and give equations of the asymptotes. Use the graphs to determine each function's domain and range.



59. $g(x) = \log(x - 1)$

60. $g(x) = \log(x - 2)$

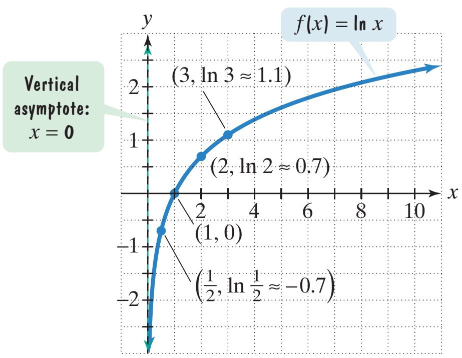
61. $h(x) = \log x - 1$

62. $h(x) = \log x - 2$

63. $g(x) = 1 - \log x$

64. $g(x) = 2 - \log x$

The figure shows the graph of $f(x) = \ln x$. In Exercises 65–74, use transformations of this graph to graph each function. Graph and give equations of the asymptotes. Use the graphs to determine each function's domain and range.



65. $g(x) = \ln(x + 2)$

66. $g(x) = \ln(x + 1)$

67. $h(x) = \ln(2x)$

68. $h(x) = \ln\left(\frac{1}{2}x\right)$

69. $g(x) = 2 \ln x$

70. $g(x) = \frac{1}{2} \ln x$

71. $h(x) = -\ln x$

72. $h(x) = \ln(-x)$

73. $g(x) = 2 - \ln x$

74. $g(x) = 1 - \ln x$

In Exercises 75–80, find the domain of each logarithmic function.

75. $f(x) = \log_5(x + 4)$

76. $f(x) = \log_5(x + 6)$

77. $f(x) = \log(2 - x)$

78. $f(x) = \log(7 - x)$

79. $f(x) = \ln(x - 2)^2$

80. $f(x) = \ln(x - 7)^2$

In Exercises 81–100, evaluate or simplify each expression without using a calculator.

81. $\log 100$

82. $\log 1000$

83. $\log 10^7$

84. $\log 10^8$

85. $10^{\log 33}$

86. $10^{\log 53}$

87. $\ln 1$

88. $\ln e$

89. $\ln e^6$

90. $\ln e^7$

91. $\ln \frac{1}{e^6}$

92. $\ln \frac{1}{e^7}$

93. $e^{\ln 125}$

94. $e^{\ln 300}$

95. $\ln e^{9x}$

96. $\ln e^{13x}$

97. $e^{\ln 5x^2}$

98. $e^{\ln 7x^2}$

99. $10^{\log \sqrt{x}}$

100. $10^{\log \sqrt[3]{x}}$

Practice Plus

In Exercises 101–104, write each equation in its equivalent exponential form. Then solve for x.

101. $\log_3(x - 1) = 2$

102. $\log_5(x + 4) = 2$

103. $\log_4 x = -3$

104. $\log_{64} x = \frac{2}{3}$

In Exercises 105–108, evaluate each expression without using a calculator.

105. $\log_3(\log_7 7)$

106. $\log_5(\log_2 32)$

107. $\log_2(\log_3 81)$

108. $\log(\ln e)$

In Exercises 109–112, find the domain of each logarithmic function.

109. $f(x) = \ln(x^2 - x - 2)$

110. $f(x) = \ln(x^2 - 4x - 12)$

111. $f(x) = \log\left(\frac{x + 1}{x - 5}\right)$

112. $f(x) = \log\left(\frac{x - 2}{x + 5}\right)$

Application Exercises

The percentage of adult height attained by a girl who is x years old can be modeled by

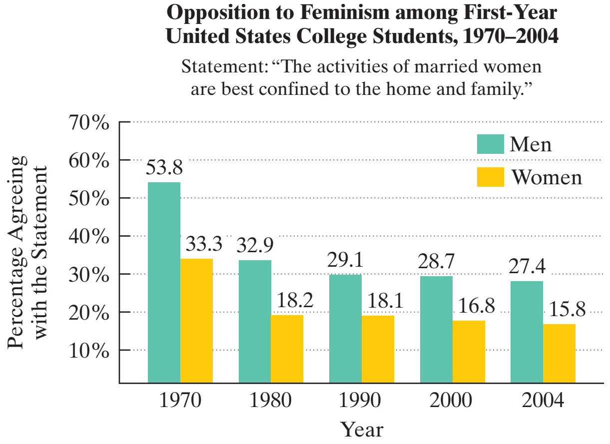
$$f(x) = 62 + 35 \log(x - 4),$$

where x represents the girl's age (from 5 to 15) and $f(x)$ represents the percentage of her adult height. Use the function to solve Exercises 113–114. Round answers to the nearest tenth of a percent.

113. Approximately what percentage of her adult height has a girl attained at age 13?

114. Approximately what percentage of her adult height has a girl attained at age ten?

The bar graph indicates that the percentage of first-year college students expressing antifeminist views has been declining since 1970. Use this information to solve Exercises 115–116.



Source: John Macionis, *Sociology,* Eleventh Edition, Prentice Hall, 2007

115. The function

$$f(x) = -7.49 \ln x + 53$$

models the percentage of first-year college men, $f(x)$, expressing antifeminist views (by agreeing with the statement) x years after 1969.

a. Use the function to find the percentage of first-year college men expressing antifeminist views in 2004. Round to one decimal place. Does this function value overestimate or underestimate the percentage displayed by the graph? By how much?

b. Use the function to project the percentage of first-year college men who will express antifeminist views in 2010. Round to one decimal place.

116. The function

$$f(x) = -4.86 \ln x + 32.5$$

models the percentage of first-year college women, $f(x)$, expressing antifeminist views (by agreeing with the statement) x years after 1969.

a. Use the function to find the percentage of first-year college women expressing antifeminist views in 2004. Round to one decimal place. Does this function value overestimate or underestimate the percentage displayed by the graph? By how much?

b. Use the function to project the percentage of first-year college women who will express antifeminist views in 2010. Round to one decimal place.

The loudness level of a sound, D, in decibels, is given by the formula

$$D = 10 \log(10^{12} I),$$

where I is the intensity of the sound, in watts per meter^2*. Decibel levels range from 0, a barely audible sound, to 160, a sound resulting in a ruptured eardrum. (Any exposure to sounds of 130 decibels or higher puts a person at immediate risk for hearing damage.) Use the formula to solve Exercises 117–118.*

117. The sound of a blue whale can be heard 500 miles away, reaching an intensity of 6.3×10^6 watts per meter^2. Determine the decibel level of this sound. At close range, can the sound of a blue whale rupture the human eardrum?

118. What is the decibel level of a normal conversation, 3.2×10^{-6} watt per meter^2?

119. Students in a psychology class took a final examination. As part of an experiment to see how much of the course content they remembered over time, they took equivalent forms of the exam in monthly intervals thereafter. The average score for the group, $f(t)$, after t months was modeled by the function

$$f(t) = 88 - 15 \ln(t + 1), \quad 0 \le t \le 12.$$

a. What was the average score on the original exam?

b. What was the average score after 2 months? 4 months? 6 months? 8 months? 10 months? one year?

c. Sketch the graph of f (either by hand or with a graphing utility). Describe what the graph indicates in terms of the material retained by the students.

Writing in Mathematics

120. Describe the relationship between an equation in logarithmic form and an equivalent equation in exponential form.

121. What question can be asked to help evaluate $\log_3 81$?

122. Explain why the logarithm of 1 with base b is 0.

123. Describe the following property using words: $\log_b b^x = x$.

124. Explain how to use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = \log_2 x$.

125. Explain how to find the domain of a logarithmic function.

126. Logarithmic models are well suited to phenomena in which growth is initially rapid but then begins to level off. Describe something that is changing over time that can be modeled using a logarithmic function.

127. Suppose that a girl is 4 feet 6 inches at age 10. Explain how to use the function in Exercises 113–114 to determine how tall she can expect to be as an adult.

Technology Exercises

In Exercises 128–131, graph f and g in the same viewing rectangle. Then describe the relationship of the graph of g to the graph of f.

128. $f(x) = \ln x, g(x) = \ln(x + 3)$

129. $f(x) = \ln x, g(x) = \ln x + 3$

130. $f(x) = \log x, g(x) = -\log x$

131. $f(x) = \log x, g(x) = \log(x - 2) + 1$

132. Students in a mathematics class took a final examination. They took equivalent forms of the exam in monthly intervals thereafter. The average score, $f(t)$, for the group after t months was modeled by the human memory function $f(t) = 75 - 10 \log(t + 1)$, where $0 \le t \le 12$. Use a graphing utility to graph the function. Then determine how many months will elapse before the average score falls below 65.

133. In parts (a)–(c), graph f and g in the same viewing rectangle.

a. $f(x) = \ln(3x), g(x) = \ln 3 + \ln x$

b. $f(x) = \log(5x^2), g(x) = \log 5 + \log x^2$

c. $f(x) = \ln(2x^3), g(x) = \ln 2 + \ln x^3$

d. Describe what you observe in parts (a)–(c). Generalize this observation by writing an equivalent expression for $\log_b(MN)$, where $M > 0$ and $N > 0$.

e. Complete this statement: The logarithm of a product is equal to ______________________.

134. Graph each of the following functions in the same viewing rectangle and then place the functions in order from the one that increases most slowly to the one that increases most rapidly.

$$y = x, y = \sqrt{x}, y = e^x, y = \ln x, y = x^x, y = x^2$$

Critical Thinking Exercises

Make Sense? *In Exercises 135–138, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

135. I've noticed that exponential functions and logarithmic functions exhibit inverse, or opposite, behavior in many ways. For example, a vertical translation shifts an exponential function's horizontal asymptote and a horizontal translation shifts a logarithmic function's vertical asymptote.

136. I estimate that $\log_8 16$ lies between 1 and 2 because $8^1 = 8$ and $8^2 = 64$.

137. I can evaluate some common logarithms without having to use a calculator.

138. An earthquake of magnitude 8 on the Richter scale is twice as intense as an earthquake of magnitude 4.

In Exercises 139–142, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

139. $\dfrac{\log_2 8}{\log_2 4} = \dfrac{8}{4}$

140. $\log(-100) = -2$

141. The domain of $f(x) = \log_2 x$ is $(-\infty, \infty)$.

142. $\log_b x$ is the exponent to which b must be raised to obtain x.

143. Without using a calculator, find the exact value of

$$\frac{\log_3 81 - \log_\pi 1}{\log_{2\sqrt{2}} 8 - \log 0.001}.$$

144. Without using a calculator, find the exact value of $\log_4[\log_3(\log_2 8)]$.

145. Without using a calculator, determine which is the greater number: $\log_4 60$ or $\log_3 40$.

Group Exercise

146. This group exercise involves exploring the way we grow. Group members should create a graph for the function that models the percentage of adult height attained by a boy who is x years old, $f(x) = 29 + 48.8 \log(x + 1)$. Let $x = 1, 2, 3, \ldots, 12$, find function values, and connect the resulting points with a smooth curve. Then create a graph for the function that models the percentage of adult height attained by a girl who is x years old, $g(x) = 62 + 35 \log(x - 4)$. Let $x = 5, 6, 7, \ldots, 15$, find function values, and connect the resulting points with a smooth curve. Group members should then discuss similarities and differences in the growth patterns for boys and girls based on the graphs.

Preview Exercises

Exercises 147–149 will help you prepare for the material covered in the next section. In each exercise, evaluate the indicated logarithmic expressions without using a calculator.

147. **a.** Evaluate: $\log_2 32$.
b. Evaluate: $\log_2 8 + \log_2 4$.
c. What can you conclude about $\log_2 32$, or $\log_2(8 \cdot 4)$?

148. **a.** Evaluate: $\log_2 16$.
b. Evaluate: $\log_2 32 - \log_2 2$.
c. What can you conclude about

$$\log_2 16, \text{ or } \log_2\left(\frac{32}{2}\right)?$$

149. **a.** Evaluate: $\log_3 81$.
b. Evaluate: $2 \log_3 9$.
c. What can you conclude about

$$\log_3 81, \text{ or } \log_3 9^2?$$

Section 3.3 Properties of Logarithms

Objectives

1. Use the product rule.
2. Use the quotient rule.
3. Use the power rule.
4. Expand logarithmic expressions.
5. Condense logarithmic expressions.
6. Use the change-of-base property.

We all learn new things in different ways. In this section, we consider important properties of logarithms. What would be the most effective way for you to learn these properties? Would it be helpful to use your graphing utility and discover one of these properties for yourself? To do so, work Exercise 133 in Exercise Set 3.2 before continuing. Would it be helpful to evaluate certain logarithmic expressions that suggest three of the properties? If this is the case, work Preview Exercises 147–149 in Exercise Set 3.2 before continuing. Would the properties become more meaningful if you could see exactly where they come from? If so, you will find details of the proofs of many of these properties in Appendix A. The remainder of our work in this chapter will be based on the properties of logarithms that you learn in this section.