119. When I convert degrees to radians, I multiply by 1, choosing $\frac{\pi}{180°}$ for 1.

120. Using radian measure, I can always find a positive angle less than 2π coterminal with a given angle by adding or subtracting 2π.

121. If $\theta = \frac{3}{2}$, is this angle larger or smaller than a right angle?

122. A railroad curve is laid out on a circle. What radius should be used if the track is to change direction by 20° in a distance of 100 miles? Round your answer to the nearest mile.

123. Assuming Earth to be a sphere of radius 4000 miles, how many miles north of the Equator is Miami, Florida, if it is 26° north from the Equator? Round your answer to the nearest mile.

Preview Exercises

Exercises 124–126 will help you prepare for the material covered in the next section.

124. Graph: $x^2 + y^2 = 1$. Then locate the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the graph.

125. Use your graph of $x^2 + y^2 = 1$ from Exercise 124 to determine the relation's domain and range.

126. Find $\frac{x}{y}$ for $x = -\frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$, and then rationalize the denominator.

Section 4.2 Trigonometric Functions: The Unit Circle

Objectives

1. Use a unit circle to define trigonometric functions of real numbers.
2. Recognize the domain and range of sine and cosine functions.
3. Find exact values of the trigonometric functions at $\frac{\pi}{4}$.
4. Use even and odd trigonometric functions.
5. Recognize and use fundamental identities.
6. Use periodic properties.
7. Evaluate trigonometric functions with a calculator.

There is something comforting in the repetition of some of nature's patterns. The ocean level at a beach varies between high and low tide approximately every 12 hours. The number of hours of daylight oscillates from a maximum on the summer solstice, June 21, to a minimum on the winter solstice, December 21. Then it increases to the same maximum the following June 21. Some believe that cycles, called biorhythms, represent physical, emotional, and intellectual aspects of our lives. In this chapter, we study six functions, the six *trigonometric functions*, that are used to model phenomena that occur again and again.

Calculus and the Unit Circle

The word *trigonometry* means measurement of triangles. Trigonometric functions, with domains consisting of sets of angles, were first defined using right triangles. By contrast, problems in calculus are solved using functions whose domains are sets of real numbers. Therefore, we introduce the trigonometric functions using unit circles and radians, rather than right triangles and degrees.

A **unit circle** is a circle of radius 1, with its center at the origin of a rectangular coordinate system. The equation of this unit circle is $x^2 + y^2 = 1$. **Figure 4.19** shows a unit circle with a central angle measuring t radians.

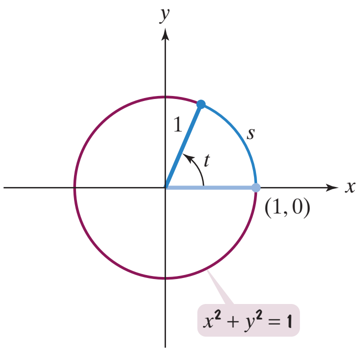


Figure 4.19 Unit circle with a central angle measuring t radians

We can use the formula for the length of a circular arc, $s = r\theta$, to find the length of the intercepted arc.

$$s = \underbrace{r}_{\text{The radius of a unit circle is 1.}}\theta = 1 \cdot \underbrace{t}_{\text{The radian measure of the central angle is } t.} = t$$

Thus, the length of the intercepted arc is t. This is also the radian measure of the central angle. Thus, **in a unit circle, the radian measure of the central angle is equal to the length of the intercepted arc**. Both are given by the same *real number* t.

In **Figure 4.20**, the radian measure of the angle and the length of the intercepted arc are both shown by t. Let $P = (x, y)$ denote the point on the unit circle that has arc length t from $(1, 0)$. **Figure 4.20(a)** shows that if t is positive, point P is reached by moving counterclockwise along the unit circle from $(1, 0)$. **Figure 4.20(b)** shows that if t is negative, point P is reached by moving clockwise along the unit circle from $(1, 0)$. For each real number t, there corresponds a point $P = (x, y)$ on the unit circle.

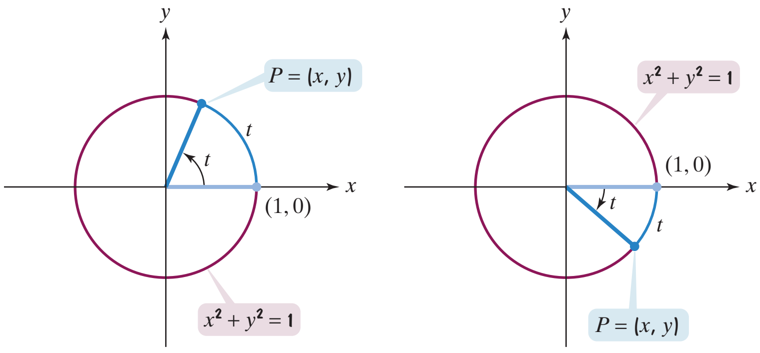


Figure 4.20 **(a)** t is positive. **(b)** t is negative.

1 Use a unit circle to define trigonometric functions of real numbers.

The Six Trigonometric Functions

We begin the study of trigonometry by defining the six trigonometric functions. The inputs of these functions are real numbers, represented by t in **Figure 4.20**. The outputs involve the point $P = (x, y)$ on the unit circle that corresponds to t and the coordinates of this point.

The trigonometric functions have names that are words, rather than single letters such as f, g, and h. For example, the **sine of *t*** is the y-coordinate of point P on the unit circle:

$$\sin t = y.$$

Input is the real number t.

Output is the y-coordinate of a point on the unit circle.

The value of y depends on the real number t and thus is a function of t. The expression $\sin t$ really means $\sin(t)$, where sine is the name of the function and t, a real number, is an input.

For example, a point $P = (x, y)$ on the unit circle corresponding to a real number t is shown in **Figure 4.21** for $\pi < t < \frac{3\pi}{2}$. We see that the coordinates of $P = (x, y)$ are $x = -\frac{3}{5}$ and $y = -\frac{4}{5}$. Because the sine function is the y-coordinate of P, the value of this trigonometric function at the real number t is

$$\sin t = -\frac{4}{5}.$$

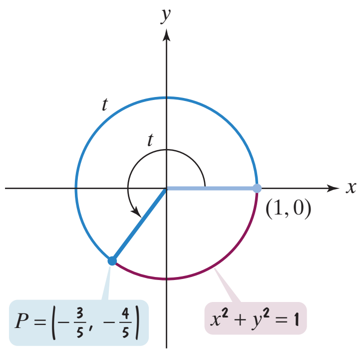


Figure 4.21

Here are the names of the six trigonometric functions, along with their abbreviations.

Name	Abbreviation	Name	Abbreviation
sine	sin	cosecant	csc
cosine	cos	secant	sec
tangent	tan	cotangent	cot

Definitions of the Trigonometric Functions in Terms of a Unit Circle

If t is a real number and $P = (x, y)$ is a point on the unit circle that corresponds to t, then

$$\sin t = y \qquad \csc t = \frac{1}{y},\ y \neq 0$$

$$\cos t = x \qquad \sec t = \frac{1}{x},\ x \neq 0$$

$$\tan t = \frac{y}{x},\ x \neq 0 \qquad \cot t = \frac{x}{y},\ y \neq 0.$$

Because this definition expresses function values in terms of coordinates of a point on a unit circle, the trigonometric functions are sometimes called the **circular functions**. Observe that the function values in the second column in the box are the reciprocals of the corresponding function values in the first column.

EXAMPLE 1 Finding Values of the Trigonometric Functions

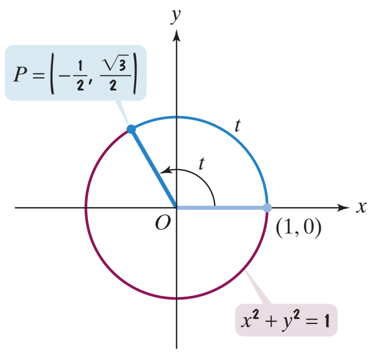


Figure 4.22

In **Figure 4.22**, t is a real number equal to the length of the intercepted arc of an angle that measures t radians and $P = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is the point on the unit circle that corresponds to t. Use the figure to find the values of the trigonometric functions at t.

Solution The point P on the unit circle that corresponds to t has coordinates $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. We use $x = -\frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ to find the values of the trigonometric functions. Because radical expressions are usually written without radicals in the denominators, we simplify by rationalizing denominators where appropriate.

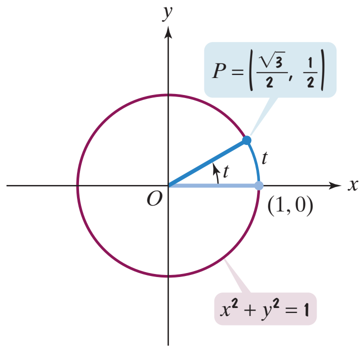
$$\sin t = y = \frac{\sqrt{3}}{2} \qquad \csc t = \frac{1}{y} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cos t = x = -\frac{1}{2} \qquad \sec t = \frac{1}{x} = \frac{1}{-\frac{1}{2}} = -2$$

Rationalize denominators. We are multiplying by 1 and not changing function values.

$$\tan t = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3} \qquad \cot t = \frac{x}{y} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Check Point 1 Use the figure on the right to find the values of the trigonometric functions at t.



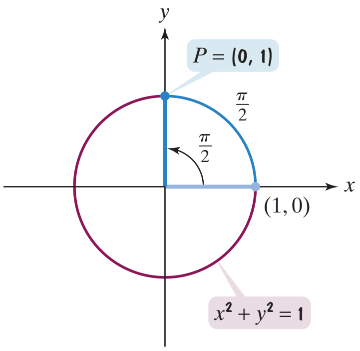


Figure 4.23

EXAMPLE 2 Finding Values of the Trigonometric Functions

Use **Figure 4.23** to find the values of the trigonometric functions at $t = \frac{\pi}{2}$.

Solution The point P on the unit circle that corresponds to $t = \frac{\pi}{2}$ has coordinates $(0, 1)$. We use $x = 0$ and $y = 1$ to find the values of the trigonometric functions at $\frac{\pi}{2}$.

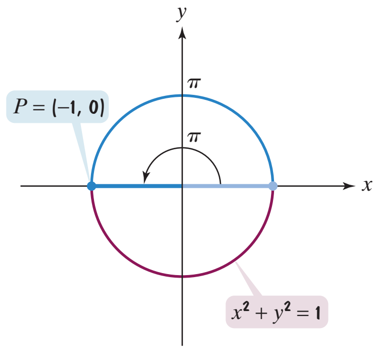
$$\sin\frac{\pi}{2} = y = 1 \qquad \csc\frac{\pi}{2} = \frac{1}{y} = \frac{1}{1} = 1$$

$$\cos\frac{\pi}{2} = x = 0 \qquad \sec\frac{\pi}{2} = \frac{1}{x} = \cancel{\frac{1}{0}}$$

$$\tan\frac{\pi}{2} = \frac{y}{x} = \cancel{\frac{1}{0}} \qquad \cot\frac{\pi}{2} = \frac{x}{y} = \frac{0}{1} = 0$$

$\sec\frac{\pi}{2}$ and $\tan\frac{\pi}{2}$ are undefined.

Check Point 2 Use the figure on the right to find the values of the trigonometric functions at $t = \pi$.



2 Recognize the domain and range of sine and cosine functions.

Domain and Range of Sine and Cosine Functions

The domain and range of each trigonometric function can be found from the unit circle definition. At this point, let's look only at the sine and cosine functions,

$$\sin t = y \quad \text{and} \quad \cos t = x.$$

Figure 4.24 shows the sine function at t as the y-coordinate of a point along the unit circle:

$$y = \sin t.$$

The domain is associated with t, the angle's radian measure and the intercepted arc's length.

The range is associated with y, the point's second coordinate.

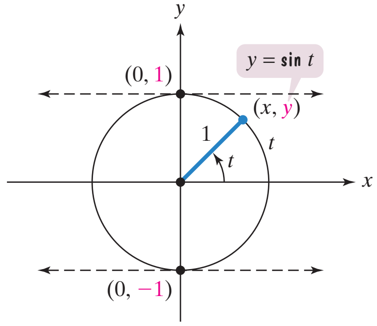


Figure 4.24

Because t can be any real number, the domain of the sine function is $(-\infty, \infty)$, the set of all real numbers. The radius of the unit circle is 1 and the dashed horizontal lines in **Figure 4.24** show that y cannot be less than -1 or greater than 1. Thus, the range of the sine function is $[-1, 1]$, the set of all real numbers from -1 to 1, inclusive.

Figure 4.25 shows the cosine function at t as the x-coordinate of a point along the unit circle:

$$x = \cos t.$$

The domain is associated with t, the angle's radian measure and the intercepted arc's length.

The range is associated with x, the point's first coordinate.

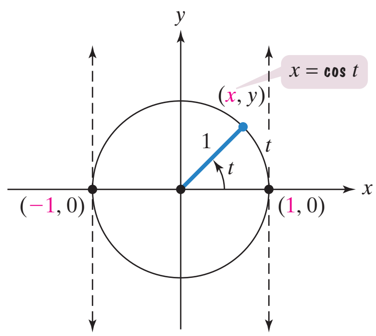


Figure 4.25

Because t can be any real number, the domain of the cosine function is $(-\infty, \infty)$. The radius of the unit circle is 1 and the dashed vertical lines in **Figure 4.25** show that x cannot be less than -1 or greater than 1. Thus, the range of the cosine function is $[-1, 1]$.

The Domain and Range of the Sine and Cosine Functions

The domain of the sine function and the cosine function is $(-\infty, \infty)$, the set of all real numbers. The range of these functions is $[-1, 1]$, the set of all real numbers from -1 to 1, inclusive.

3 Find exact values of the trigonometric functions at $\frac{\pi}{4}$.

Exact Values of the Trigonometric Functions at $t = \frac{\pi}{4}$

Trigonometric functions at $t = \frac{\pi}{4}$ occur frequently. How do we use the unit circle to find values of the trigonometric functions at $t = \frac{\pi}{4}$? Look at **Figure 4.26**. We must find the coordinates of point $P = (a, b)$ on the unit circle that correspond to $t = \frac{\pi}{4}$. Can you see that P lies on the line $y = x$? Thus, point P has equal x- and y-coordinates: $a = b$. We find these coordinates as follows:

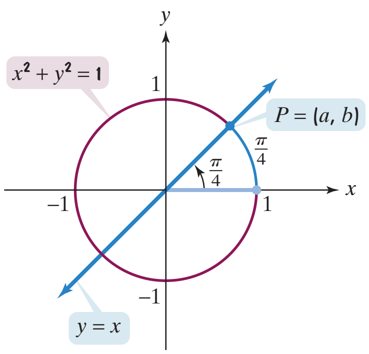


Figure 4.26

$x^2 + y^2 = 1$ — This is the equation of the unit circle.

$a^2 + b^2 = 1$ — Point $P = (a, b)$ lies on the unit circle. Thus, its coordinates satisfy the circle's equation.

$a^2 + a^2 = 1$ — Because $a = b$, substitute a for b in the previous equation.

$2a^2 = 1$ — Add like terms.

$a^2 = \frac{1}{2}$ — Divide both sides of the equation by 2.

$a = \sqrt{\frac{1}{2}}$ — Because $a > 0$, take the positive square root of both sides.

We see that $a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$. Because $a = b$, we also have $b = \frac{1}{\sqrt{2}}$. Thus, if $t = \frac{\pi}{4}$, point $P = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the point on the unit circle that corresponds to t. Let's rationalize the denominator on each coordinate:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

We are multiplying by 1 and not changing the value of $\frac{1}{\sqrt{2}}$.

We use $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ to find the values of the trigonometric functions at $t = \frac{\pi}{4}$.

EXAMPLE 3 Finding Values of the Trigonometric Functions at $t = \frac{\pi}{4}$

Find $\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$, and $\tan\frac{\pi}{4}$.

Solution The point P on the unit circle that corresponds to $t = \frac{\pi}{4}$ has coordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. We use $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$ to find the values of the three trigonometric functions at $\frac{\pi}{4}$.

$$\sin\frac{\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = x = \frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4} = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

Check Point 3 Find $\csc\frac{\pi}{4}$, $\sec\frac{\pi}{4}$, and $\cot\frac{\pi}{4}$.

Because you will often see the trigonometric functions at $\frac{\pi}{4}$, it is a good idea to memorize the values shown in the following box. In the next section, you will learn to use a right triangle to obtain these values.

Trigonometric Functions at $\frac{\pi}{4}$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \csc\frac{\pi}{4} = \sqrt{2}$$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \cot\frac{\pi}{4} = 1$$

4 Use even and odd trigonometric functions.

Even and Odd Trigonometric Functions

We have seen that a function is even if $f(-t) = f(t)$ and odd if $f(-t) = -f(t)$. We can use **Figure 4.27** to show that the cosine function is an even function and the sine function is an odd function. By definition, the coordinates of the points P and Q in **Figure 4.27** are as follows:

$$P: (\cos t, \sin t)$$

$$Q: (\cos(-t), \sin(-t)).$$

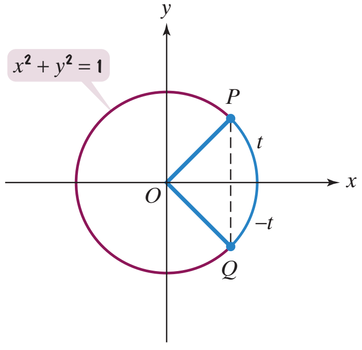


Figure 4.27

In **Figure 4.27**, the x-coordinates of P and Q are the same. Thus,

$$\cos(-t) = \cos t.$$

This shows that the cosine function is an even function. By contrast, the y-coordinates of P and Q are negatives of each other. Thus,

$$\sin(-t) = -\sin t.$$

This shows that the sine function is an odd function.

This argument is valid regardless of the length of t. Thus, the arc may terminate in any of the four quadrants or on any axis. Using the unit circle definition of the trigonometric functions, we obtain the following results:

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$$
$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

EXAMPLE 4 Using Even and Odd Functions to Find Values of Trigonometric Functions

Find the value of each trigonometric function:

a. $\cos\left(-\frac{\pi}{4}\right)$ **b.** $\tan\left(-\frac{\pi}{4}\right)$.

Solution

a. $\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ **b.** $\tan\left(-\frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$

Check Point 4 Find the value of each trigonometric function:

a. $\sec\left(-\frac{\pi}{4}\right)$ **b.** $\sin\left(-\frac{\pi}{4}\right)$.

5 Recognize and use fundamental identities.

Fundamental Identities

Many relationships exist among the six trigonometric functions. These relationships are described using **trigonometric identities**. Trigonometric identities are equations that are true for all real numbers for which the trigonometric expressions in the equations are defined. For example, the definitions of the cosine and secant functions are given by

$$\cos t = x \quad \text{and} \quad \sec t = \frac{1}{x},\ x \neq 0.$$

Substituting $\cos t$ for x in the equation on the right, we see that

$$\sec t = \frac{1}{\cos t},\ \cos t \neq 0.$$

This identity is one of six **reciprocal identities**.

Reciprocal Identities

$$\sin t = \frac{1}{\csc t} \qquad \csc t = \frac{1}{\sin t}$$
$$\cos t = \frac{1}{\sec t} \qquad \sec t = \frac{1}{\cos t}$$
$$\tan t = \frac{1}{\cot t} \qquad \cot t = \frac{1}{\tan t}$$

Two other relationships that follow from the definitions of the trigonometric functions are called the **quotient identities**.

Quotient Identities

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

If $\sin t$ and $\cos t$ are known, a quotient identity and three reciprocal identities make it possible to find the value of each of the four remaining trigonometric functions.

EXAMPLE 5 Using Quotient and Reciprocal Identities

Given $\sin t = \frac{2}{5}$ and $\cos t = \frac{\sqrt{21}}{5}$, find the value of each of the four remaining trigonometric functions.

Solution We can find $\tan t$ by using the quotient identity that describes $\tan t$ as the quotient of $\sin t$ and $\cos t$.

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{2}{5}}{\frac{\sqrt{21}}{5}} = \frac{2}{5} \cdot \frac{5}{\sqrt{21}} = \frac{2}{\sqrt{21}} = \frac{2}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

Rationalize the denominator.

We use the reciprocal identities to find the value of each of the remaining three functions.

$$\csc t = \frac{1}{\sin t} = \frac{1}{\frac{2}{5}} = \frac{5}{2}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{\sqrt{21}}{5}} = \frac{5}{\sqrt{21}} = \frac{5}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = \frac{5\sqrt{21}}{21}$$

Rationalize the denominator.

$$\cot t = \frac{1}{\tan t} = \frac{1}{\frac{2}{\sqrt{21}}} = \frac{\sqrt{21}}{2}$$

We found $\tan t = \frac{2}{\sqrt{21}}$. We could use $\tan t = \frac{2\sqrt{21}}{21}$, but then we would have to rationalize the denominator.

Check Point 5 Given $\sin t = \frac{2}{3}$ and $\cos t = \frac{\sqrt{5}}{3}$, find the value of each of the four remaining trigonometric functions.

Other relationships among trigonometric functions follow from the equation of the unit circle

$$x^2 + y^2 = 1.$$

Because $\cos t = x$ and $\sin t = y$, we see that

$$(\cos t)^2 + (\sin t)^2 = 1.$$

We will eliminate the parentheses in this identity by writing $\cos^2 t$ instead of $(\cos t)^2$ and $\sin^2 t$ instead of $(\sin t)^2$. With this notation, we can write the identity as

$$\cos^2 t + \sin^2 t = 1$$

or

$$\sin^2 t + \cos^2 t = 1. \quad \text{The identity usually appears in this form.}$$

Two additional identities can be obtained from $x^2 + y^2 = 1$ by dividing both sides by x^2 and y^2, respectively. The three identities are called the **Pythagorean identities**.

Pythagorean Identities

$$\sin^2 t + \cos^2 t = 1 \qquad 1 + \tan^2 t = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t$$

EXAMPLE 6 Using a Pythagorean Identity

Given that $\sin t = \frac{3}{5}$ and $0 \le t < \frac{\pi}{2}$, find the value of $\cos t$ using a trigonometric identity.

Solution We can find the value of $\cos t$ by using the Pythagorean identity

$$\sin^2 t + \cos^2 t = 1.$$

$$\left(\frac{3}{5}\right)^2 + \cos^2 t = 1 \qquad \text{We are given that } \sin t = \frac{3}{5}.$$

$$\frac{9}{25} + \cos^2 t = 1 \qquad \text{Square } \frac{3}{5}\text{: } \left(\frac{3}{5}\right)^2 = \frac{3^2}{5^2} = \frac{9}{25}.$$

$$\cos^2 t = 1 - \frac{9}{25} \qquad \text{Subtract } \frac{9}{25} \text{ from both sides.}$$

$$\cos^2 t = \frac{16}{25} \qquad \text{Simplify: } 1 - \frac{9}{25} = \frac{25}{25} - \frac{9}{25} = \frac{16}{25}.$$

$$\cos t = \sqrt{\frac{16}{25}} = \frac{4}{5} \qquad \text{Because } 0 \le t < \frac{\pi}{2}, \cos t \text{, the x-coordinate of a point on the unit circle, is positive.}$$

Thus, $\cos t = \frac{4}{5}$.

Check Point 6 Given that $\sin t = \frac{1}{2}$ and $0 \le t < \frac{\pi}{2}$, find the value of $\cos t$ using a trigonometric identity.

6 Use periodic properties.

Periodic Functions

Certain patterns in nature repeat again and again. For example, the ocean level at a beach varies from low tide to high tide and then back to low tide approximately every 12 hours. If low tide occurs at noon, then high tide will be around 6 P.M. and low tide will occur again around midnight, and so on infinitely. If $f(t)$ represents the ocean level at the beach at any time t, then the level is the same 12 hours later. Thus,

$$f(t + 12) = f(t).$$

The word *periodic* means that this tidal behavior repeats infinitely. The *period*, 12 hours, is the time it takes to complete one full cycle.

Definition of a Periodic Function

A function f is **periodic** if there exists a positive number p such that

$$f(t + p) = f(t)$$

for all t in the domain of f. The smallest positive number p for which f is periodic is called the **period** of f.

The trigonometric functions are used to model periodic phenomena. Why? If we begin at any point P on the unit circle and travel a distance of 2π units along the perimeter, we will return to the same point P. Because the trigonometric functions are defined in terms of the coordinates of that point P, we obtain the following results:

Periodic Properties of the Sine and Cosine Functions

$$\sin(t + 2\pi) = \sin t \quad \text{and} \quad \cos(t + 2\pi) = \cos t$$

The sine and cosine functions are periodic functions and have period 2π.

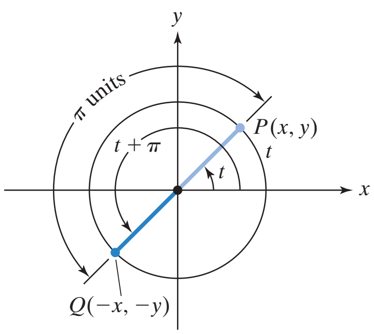


Figure 4.28
tangent at P = tangent at Q

Like the sine and cosine functions, the secant and cosecant functions have period 2π. However, the tangent and cotangent functions have a smaller period. **Figure 4.28** shows that if we begin at any point $P(x, y)$ on the unit circle and travel a distance of π units along the perimeter, we arrive at the point $Q(-x, -y)$. The tangent function, defined in terms of the coordinates of a point, is the same at (x, y) and $(-x, -y)$.

Tangent function at (x, y) $\quad \dfrac{y}{x} = \dfrac{-y}{-x} \quad$ Tangent function π radians later

We see that $\tan(t + \pi) = \tan t$. The same observations apply to the cotangent function.

Periodic Properties of the Tangent and Cotangent Functions

$$\tan(t + \pi) = \tan t \quad \text{and} \quad \cot(t + \pi) = \cot t$$

The tangent and cotangent functions are periodic functions and have period π.

EXAMPLE 7 Using Periodic Properties

Find the value of each trigonometric function:

a. $\sin \dfrac{9\pi}{4}$ **b.** $\tan\left(-\dfrac{5\pi}{4}\right)$.

Solution

a. $$\sin \frac{9\pi}{4} = \sin\left(\frac{\pi}{4} + 2\pi\right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$\sin(t + 2\pi) = \sin t$

b. $$\tan\left(-\frac{5\pi}{4}\right) = -\tan \frac{5\pi}{4} = -\tan\left(\frac{\pi}{4} + \pi\right) = -\tan \frac{\pi}{4} = -1$$

The tangent function is odd: $\tan(-t) = -\tan t$.

$\tan(t + \pi) = \tan t$

Check Point 7 Find the value of each trigonometric function:

a. $\cot \dfrac{5\pi}{4}$ **b.** $\cos\left(-\dfrac{9\pi}{4}\right)$.

Why do the trigonometric functions model phenomena that repeat *indefinitely*? By starting at point P on the unit circle and traveling a distance of 2π units, 4π units, 6π units, and so on, we return to the starting point P. Because the trigonometric functions are defined in terms of the coordinates of that point P, if we add (or subtract) multiples of 2π to t, the values of the trigonometric functions of t

do not change. Furthermore, the values for the tangent and cotangent functions of t do not change if we add (or subtract) multiples of π to t.

Repetitive Behavior of the Sine, Cosine, and Tangent Functions

For any integer n and real number t,

$$\sin(t + 2\pi n) = \sin t, \quad \cos(t + 2\pi n) = \cos t, \quad \text{and} \quad \tan(t + \pi n) = \tan t.$$

7 Evaluate trigonometric functions with a calculator.

Using a Calculator to Evaluate Trigonometric Functions

We used a unit circle to find values of the trigonometric functions at $\frac{\pi}{4}$. These are exact values. We can find approximate values of the trigonometric functions using a calculator.

The first step in using a calculator to evaluate trigonometric functions is to set the calculator to the correct *mode*, degrees or radians. The domains of the trigonometric functions in the unit circle are sets of real numbers. Therefore, we use the radian mode.

Most calculators have keys marked $\boxed{\text{SIN}}$, $\boxed{\text{COS}}$, and $\boxed{\text{TAN}}$. For example, to find the value of sin 1.2, set the calculator to the radian mode and enter 1.2 $\boxed{\text{SIN}}$ on most scientific calculators and $\boxed{\text{SIN}}$ 1.2 $\boxed{\text{ENTER}}$ on most graphing calculators. Consult the manual for your calculator.

To evaluate the cosecant, secant, and cotangent functions, use the key for the respective reciprocal function, $\boxed{\text{SIN}}$, $\boxed{\text{COS}}$, or $\boxed{\text{TAN}}$, and then use the reciprocal key. The reciprocal key is $\boxed{1/x}$ on many scientific calculators and $\boxed{x^{-1}}$ on many graphing calculators. For example, we can evaluate $\sec \frac{\pi}{12}$ using the following reciprocal relationship:

$$\sec \frac{\pi}{12} = \frac{1}{\cos \frac{\pi}{12}}.$$

Using the radian mode, enter one of the following keystroke sequences:

Many Scientific Calculators

$$\boxed{\pi}\ \boxed{\div}\ 12\ \boxed{=}\ \boxed{\text{COS}}\ \boxed{1/x}$$

Many Graphing Calculators

$$\boxed{(}\ \boxed{\text{COS}}\ \boxed{(}\ \boxed{\pi}\ \boxed{\div}\ 12\ \boxed{)}\ \boxed{)}\ \boxed{x^{-1}}\ \boxed{\text{ENTER}}.$$

(The open parenthesis following the $\boxed{\text{COS}}$ key is provided on some calculators.) Rounding the display to four decimal places, we obtain $\sec \frac{\pi}{12} \approx 1.0353$.

EXAMPLE 8 Evaluating Trigonometric Functions with a Calculator

Use a calculator to find the value to four decimal places:

a. $\cos \frac{\pi}{4}$ **b.** cot 1.2.

Solution

Scientific Calculator Solution

Function	Mode	Keystrokes	Display, rounded to four decimal places
a. $\cos \frac{\pi}{4}$	Radian	$\boxed{\pi}\ \boxed{\div}\ 4\ \boxed{=}\ \boxed{\text{COS}}$	0.7071
b. cot 1.2	Radian	1.2 $\boxed{\text{TAN}}\ \boxed{1/x}$	0.3888

Graphing Calculator Solution

	Function	Mode	Keystrokes	Display, rounded to four decimal places
a.	$\cos \frac{\pi}{4}$	Radian	$\boxed{\text{COS}}\ \boxed{(}\ \boxed{\pi}\ \boxed{\div}\ 4\ \boxed{)}\ \boxed{\text{ENTER}}$	0.7071
b.	$\cot 1.2$	Radian	$\boxed{(}\ \boxed{\text{TAN}}\ 1.2\ \boxed{)}\ \boxed{x^{-1}}\ \boxed{\text{ENTER}}$	0.3888

Check Point 8 Use a calculator to find the value to four decimal places:

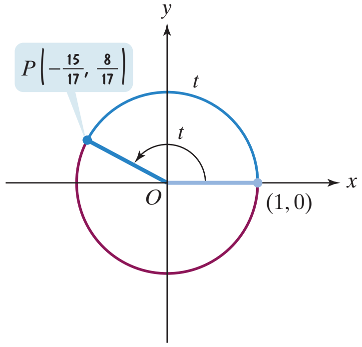
a. $\sin \frac{\pi}{4}$ **b.** $\csc 1.5$.

Exercise Set 4.2

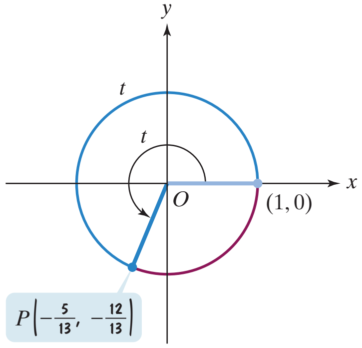
Practice Exercises

In Exercises 1–4, a point $P(x, y)$ is shown on the unit circle corresponding to a real number t. Find the values of the trigonometric functions at t.

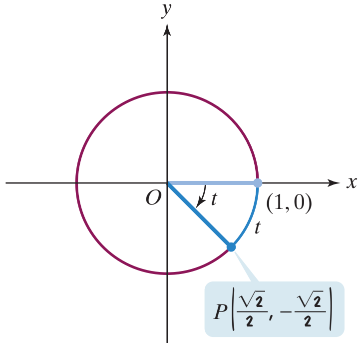
1.



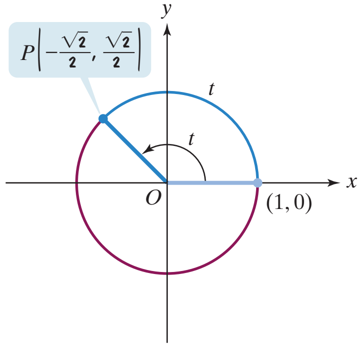
2.



3.



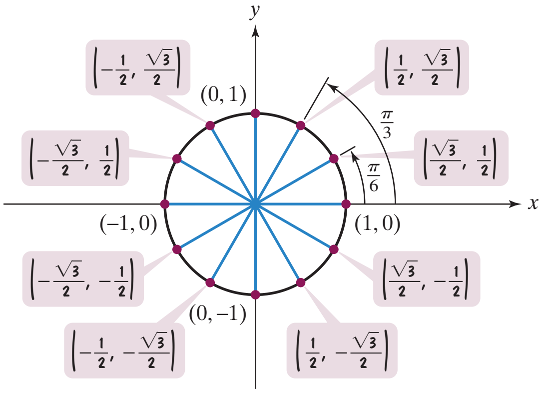
4.



In Exercises 5–18, the unit circle has been divided into twelve equal arcs, corresponding to t-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$

Use the (x, y) coordinates in the figure to find the value of each trigonometric function at the indicated real number, t, or state that the expression is undefined.



5. $\sin \frac{\pi}{6}$ **6.** $\sin \frac{\pi}{3}$ **7.** $\cos \frac{5\pi}{6}$

8. $\cos \frac{2\pi}{3}$ **9.** $\tan \pi$ **10.** $\tan 0$

11. $\csc \frac{7\pi}{6}$ **12.** $\csc \frac{4\pi}{3}$ **13.** $\sec \frac{11\pi}{6}$

14. $\sec \frac{5\pi}{3}$ **15.** $\sin \frac{3\pi}{2}$ **16.** $\cos \frac{3\pi}{2}$

17. $\sec \frac{3\pi}{2}$ **18.** $\tan \frac{3\pi}{2}$

In Exercises 19–24,

a. *Use the unit circle shown for Exercises 5–18 to find the value of the trigonometric function.*

b. *Use even and odd properties of trigonometric functions and your answer from part (a) to find the value of the same trigonometric function at the indicated real number.*

19. a. $\cos \frac{\pi}{6}$ **b.** $\cos\left(-\frac{\pi}{6}\right)$

20. a. $\cos \frac{\pi}{3}$ **b.** $\cos\left(-\frac{\pi}{3}\right)$

21. a. $\sin \frac{5\pi}{6}$ **b.** $\sin\left(-\frac{5\pi}{6}\right)$

22. a. $\sin \frac{2\pi}{3}$ **b.** $\sin\left(-\frac{2\pi}{3}\right)$

23. a. $\tan \frac{5\pi}{3}$ **b.** $\tan\left(-\frac{5\pi}{3}\right)$

24. a. $\tan \frac{11\pi}{6}$ **b.** $\tan\left(-\frac{11\pi}{6}\right)$

In Exercises 25–28, $\sin t$ *and* $\cos t$ *are given. Use identities to find* $\tan t$, $\csc t$, $\sec t$, *and* $\cot t$. *Where necessary, rationalize denominators.*

25. $\sin t = \frac{8}{17}, \cos t = \frac{15}{17}$

26. $\sin t = \frac{3}{5}, \cos t = \frac{4}{5}$

27. $\sin t = \frac{1}{3}, \cos t = \frac{2\sqrt{2}}{3}$

28. $\sin t = \frac{2}{3}, \cos t = \frac{\sqrt{5}}{3}$

In Exercises 29–32, $0 \le t < \frac{\pi}{2}$ *and* $\sin t$ *is given. Use the Pythagorean identity* $\sin^2 t + \cos^2 t = 1$ *to find* $\cos t$.

29. $\sin t = \frac{6}{7}$ **30.** $\sin t = \frac{7}{8}$

31. $\sin t = \frac{\sqrt{39}}{8}$ **32.** $\sin t = \frac{\sqrt{21}}{5}$

In Exercises 33–38, use an identity to find the value of each expression. Do not use a calculator.

33. $\sin 1.7 \csc 1.7$ **34.** $\cos 2.3 \sec 2.3$

35. $\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{6}$ **36.** $\sin^2 \frac{\pi}{3} + \cos^2 \frac{\pi}{3}$

37. $\sec^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{3}$ **38.** $\csc^2 \frac{\pi}{6} - \cot^2 \frac{\pi}{6}$

In Exercises 39–52, find the exact value of each trigonometric function. Do not use a calculator.

39. $\cos \frac{9\pi}{4}$ **40.** $\csc \frac{9\pi}{4}$

41. $\sin\left(-\frac{9\pi}{4}\right)$ **42.** $\sec\left(-\frac{9\pi}{4}\right)$

43. $\tan \frac{5\pi}{4}$ **44.** $\cot \frac{5\pi}{4}$

45. $\cot\left(-\frac{5\pi}{4}\right)$ **46.** $\tan\left(-\frac{9\pi}{4}\right)$

47. $-\tan\left(\frac{\pi}{4} + 15\pi\right)$ **48.** $-\cot\left(\frac{\pi}{4} + 17\pi\right)$

49. $\sin\left(-\frac{\pi}{4} - 1000\pi\right)$ **50.** $\sin\left(-\frac{\pi}{4} - 2000\pi\right)$

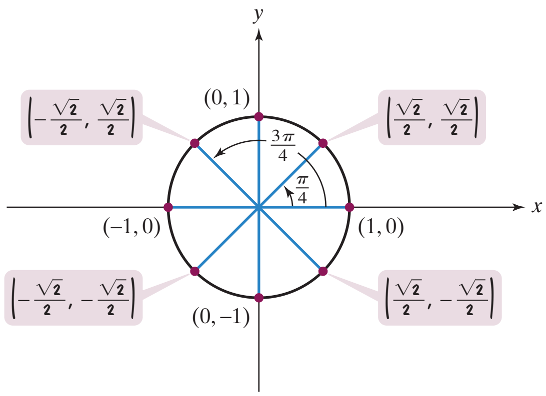
51. $\cos\left(-\frac{\pi}{4} - 1000\pi\right)$ **52.** $\cos\left(-\frac{\pi}{4} - 2000\pi\right)$

In Exercises 53–60, the unit circle has been divided into eight equal arcs, corresponding to t-values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

a. *Use the* (x, y) *coordinates in the figure to find the value of the trigonometric function.*

b. *Use periodic properties and your answer from part (a) to find the value of the same trigonometric function at the indicated real number.*



53. a. $\sin \frac{3\pi}{4}$ **b.** $\sin \frac{11\pi}{4}$

54. a. $\cos \frac{3\pi}{4}$ **b.** $\cos \frac{11\pi}{4}$

55. a. $\cos \frac{\pi}{2}$ **b.** $\cos \frac{9\pi}{2}$

56. a. $\sin \frac{\pi}{2}$ **b.** $\sin \frac{9\pi}{2}$

57. a. $\tan \pi$ **b.** $\tan 17\pi$

58. a. $\cot \frac{\pi}{2}$ **b.** $\cot \frac{15\pi}{2}$

59. a. $\sin \frac{7\pi}{4}$ **b.** $\sin \frac{47\pi}{4}$

60. a. $\cos \frac{7\pi}{4}$ **b.** $\cos \frac{47\pi}{4}$

In Exercises 61–70, use a calculator to find the value of the trigonometric function to four decimal places.

61. $\sin 0.8$ **62.** $\cos 0.6$

63. $\tan 3.4$ **64.** $\tan 3.7$

65. $\csc 1$ **66.** $\sec 1$

67. $\cos \frac{\pi}{10}$ **68.** $\sin \frac{3\pi}{10}$

69. $\cot \frac{\pi}{12}$ **70.** $\cot \frac{\pi}{18}$

Practice Plus

In Exercises 71–80, let

$$\sin t = a, \cos t = b, \text{ and } \tan t = c.$$

Write each expression in terms of a, b, and c.

71. $\sin(-t) - \sin t$

72. $\tan(-t) - \tan t$

73. $4\cos(-t) - \cos t$

74. $3\cos(-t) - \cos t$

75. $\sin(t + 2\pi) - \cos(t + 4\pi) + \tan(t + \pi)$

76. $\sin(t + 2\pi) + \cos(t + 4\pi) - \tan(t + \pi)$

77. $\sin(-t - 2\pi) - \cos(-t - 4\pi) - \tan(-t - \pi)$

78. $\sin(-t - 2\pi) + \cos(-t - 4\pi) - \tan(-t - \pi)$

79. $\cos t + \cos(t + 1000\pi) - \tan t - \tan(t + 999\pi) - \sin t + 4\sin(t - 1000\pi)$

80. $-\cos t + 7\cos(t + 1000\pi) + \tan t + \tan(t + 999\pi) + \sin t + \sin(t - 1000\pi)$

Application Exercises

81. The number of hours of daylight, H, on day t of any given year (on January 1, $t = 1$) in Fairbanks, Alaska, can be modeled by the function

$$H(t) = 12 + 8.3\sin\left[\frac{2\pi}{365}(t - 80)\right].$$

a. March 21, the 80th day of the year, is the spring equinox. Find the number of hours of daylight in Fairbanks on this day.

b. June 21, the 172nd day of the year, is the summer solstice, the day with the maximum number of hours of daylight. To the nearest tenth of an hour, find the number of hours of daylight in Fairbanks on this day.

c. December 21, the 355th day of the year, is the winter solstice, the day with the minimum number of hours of daylight. Find, to the nearest tenth of an hour, the number of hours of daylight in Fairbanks on this day.

82. The number of hours of daylight, H, on day t of any given year (on January 1, $t = 1$) in San Diego, California, can be modeled by the function

$$H(t) = 12 + 2.4\sin\left[\frac{2\pi}{365}(t - 80)\right].$$

a. March 21, the 80th day of the year, is the spring equinox. Find the number of hours of daylight in San Diego on this day.

b. June 21, the 172nd day of the year, is the summer solstice, the day with the maximum number of hours of daylight. Find, to the nearest tenth of an hour, the number of hours of daylight in San Diego on this day.

c. December 21, the 355th day of the year, is the winter solstice, the day with the minimum number of hours of daylight. To the nearest tenth of an hour, find the number of hours of daylight in San Diego on this day.

83. People who believe in biorhythms claim that there are three cycles that rule our behavior—the physical, emotional, and mental. Each is a sine function of a certain period. The function for our emotional fluctuations is

$$E = \sin\frac{\pi}{14}t,$$

where t is measured in days starting at birth. Emotional fluctuations, E, are measured from -1 to 1, inclusive, with 1 representing peak emotional well-being, -1 representing the low for emotional well-being, and 0 representing feeling neither emotionally high nor low.

a. Find E corresponding to $t = 7$, 14, 21, 28, and 35. Describe what you observe.

b. What is the period of the emotional cycle?

84. The height of the water, H, in feet, at a boat dock t hours after 6 A.M. is given by

$$H = 10 + 4\sin\frac{\pi}{6}t.$$

a. Find the height of the water at the dock at 6 A.M., 9 A.M., noon, 6 P.M., midnight, and 3 A.M.

b. When is low tide and when is high tide?

c. What is the period of this function and what does this mean about the tides?

Writing in Mathematics

85. Why are the trigonometric functions sometimes called circular functions?

86. Define the sine of t.

87. Given a point on the unit circle that corresponds to t, explain how to find $\tan t$.

88. What is the range of the sine function? Use the unit circle to explain where this range comes from.

89. Explain how to use the unit circle to find values of the trigonometric functions at $\frac{\pi}{4}$.

90. What do we mean by even trigonometric functions? Which of the six functions fall into this category?

91. Use words (not an equation) to describe one of the reciprocal identities.

92. Use words (not an equation) to describe one of the quotient identities.

93. Use words (not an equation) to describe one of the Pythagorean identities

94. What is a periodic function? Why are the sine and cosine functions periodic?

95. Explain how you can use the function for emotional fluctuations in Exercise 83 to determine good days for having dinner with your moody boss.

96. Describe a phenomenon that repeats infinitely. What is its period?

Critical Thinking Exercises

Make Sense? *In Exercises 97–100, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

97. Assuming that the innermost circle on this Navajo sand painting is a unit circle, as A moves around the circle, its coordinates define the cosine and sine functions, respectively.

98. I'm using a value for t and a point on the unit circle corresponding to t for which $\sin t = -\frac{\sqrt{10}}{2}$.

99. Because $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, I can conclude that $\cos\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

100. I can rewrite $\tan t$ as $\frac{1}{\cot t}$, as well as $\frac{\sin t}{\cos t}$.

101. If $\pi < t < \frac{3\pi}{2}$, which of the following is true?

a. $\sin t > 0$ and $\tan t > 0$.
b. $\sin t < 0$ and $\tan t < 0$.
c. $\tan t > 0$ and $\cot t > 0$.
d. $\tan t < 0$ and $\cot t < 0$.

102. If $f(x) = \sin x$ and $f(a) = \frac{1}{4}$, find the value of

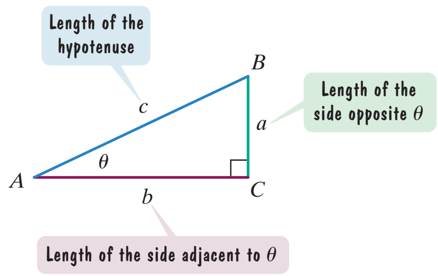
$$f(a) + f(a + 2\pi) + f(a + 4\pi) + f(a + 6\pi).$$

103. If $f(x) = \sin x$ and $f(a) = \frac{1}{4}$, find the value of $f(a) + 2f(-a)$.

104. The seats of a Ferris wheel are 40 feet from the wheel's center. When you get on the ride, your seat is 5 feet above the ground. How far above the ground are you after rotating through an angle of $\frac{17\pi}{4}$ radians? Round to the nearest foot.

Preview Exercises

Exercises 105–107 will help you prepare for the material covered in the next section. In each exercise, let θ be an acute angle in a right triangle, as shown in the figure. These exercises require the use of the Pythagorean Theorem.



105. If $a = 5$ and $b = 12$, find the ratio of the length of the side opposite θ to the length of the hypotenuse.

106. If $a = 1$ and $b = 1$, find the ratio of the length of the side opposite θ to the length of the hypotenuse. Simplify the ratio by rationalizing the denominator.

107. Simplify: $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$.

Section 4.3 Right Triangle Trigonometry

Objectives

1. Use right triangles to evaluate trigonometric functions.
2. Find function values for $30°\left(\frac{\pi}{6}\right)$, $45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.
3. Use equal cofunctions of complements.
4. Use right triangle trigonometry to solve applied problems.

In the last century, Ang Rita Sherpa climbed Mount Everest ten times, all without the use of bottled oxygen.

Mountain climbers have forever been fascinated by reaching the top of Mount Everest, sometimes with tragic results. The mountain, on Asia's Tibet-Nepal border, is Earth's highest, peaking at an incredible 29,035 feet. The heights of mountains can be found using trigonometric functions. Remember that the word "*trigonometry*" means "*measurement of triangles*." Trigonometry is used in navigation, building, and engineering. For centuries, Muslims used trigonometry and the stars to navigate across the Arabian desert to Mecca, the birthplace of the prophet Muhammad, the founder of Islam. The ancient Greeks used trigonometry to record the locations of thousands of stars and worked out the motion of the Moon relative to Earth. Today, trigonometry is used to study the structure of DNA, the master molecule that determines how we grow from a single cell to a complex, fully developed adult.

1 Use right triangles to evaluate trigonometric functions.

Right Triangle Definitions of Trigonometric Functions

We have seen that in a unit circle, the radian measure of a central angle is equal to the measure of the intercepted arc. Thus, the value of a trigonometric function at the real number t is its value at an angle of t radians.