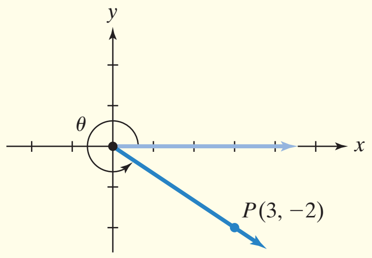
10. Use the point on the terminal side of θ to find each of the six trigonometric functions of θ.



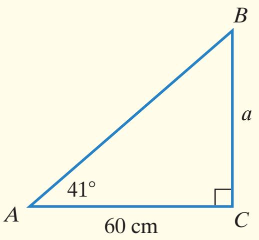
In Exercises 11–12, find the exact value of the remaining trigonometric functions of θ.

11. $\tan\theta = -\frac{3}{4}$, $\cos\theta < 0$

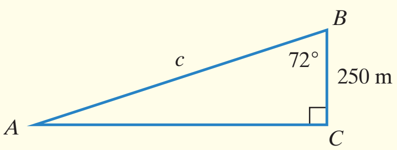
12. $\cos\theta = \frac{3}{7}$, $\sin\theta < 0$

In Exercises 13–14, find the measure of the side of the right triangle whose length is designated by a lowercase letter. Round the answer to the nearest whole number.

13.



14.



15. If $\cos\theta = \frac{1}{6}$ and θ is acute, find $\cot\left(\frac{\pi}{2} - \theta\right)$.

In Exercises 16–26, find the exact value of each expression. Do not use a calculator.

16. $\tan 30°$

17. $\cot 120°$

18. $\cos 240°$

19. $\sec \frac{11\pi}{6}$

20. $\sin^2 \frac{\pi}{7} + \cos^2 \frac{\pi}{7}$

21. $\sin\left(-\frac{2\pi}{3}\right)$

22. $\csc\left(\frac{22\pi}{3}\right)$

23. $\cos 495°$

24. $\tan\left(-\frac{17\pi}{6}\right)$

25. $\sin^2 \frac{\pi}{2} - \cos \pi$

26. $\cos\left(\frac{5\pi}{6} + 2\pi n\right) + \tan\left(\frac{5\pi}{6} + n\pi\right)$, n is an integer.

27. A circle has a radius of 40 centimeters. Find the length of the arc intercepted by a central angle of 36°. Express the answer in terms of π. Then round to two decimal places.

28. A merry-go-round makes 8 revolutions per minute. Find the linear speed, in feet per minute, of a horse 10 feet from the center. Express the answer in terms of π. Then round to one decimal place.

29. A plane takes off at an angle of 6°. After traveling for one mile, or 5280 feet, along this flight path, find the plane's height, to the nearest tenth of a foot, above the ground.

30. A tree that is 50 feet tall casts a shadow that is 60 feet long. Find the angle of elevation, to the nearest degree, of the sun.

Section 4.5 Graphs of Sine and Cosine Functions

Objectives

1. Understand the graph of $y = \sin x$.
2. Graph variations of $y = \sin x$.
3. Understand the graph of $y = \cos x$.
4. Graph variations of $y = \cos x$.
5. Use vertical shifts of sine and cosine curves.
6. Model periodic behavior.

Take a deep breath and relax. Many relaxation exercises involve slowing down our breathing. Some people suggest that the way we breathe affects every part of our lives. Did you know that graphs of trigonometric functions can be used to analyze the breathing cycle, which is our closest link to both life and death?

In this section, we use graphs of sine and cosine functions to visualize their properties. We use the traditional symbol x, rather than θ or t, to represent the independent variable. We use the symbol y for the dependent variable, or the function's value at x. Thus, we will be graphing $y = \sin x$ and $y = \cos x$ in rectangular coordinates. In all graphs of trigonometric functions, the independent variable, x, is measured in radians.

1 Understand the graph of $y = \sin x$.

The Graph of $y = \sin x$

The trigonometric functions can be graphed in a rectangular coordinate system by plotting points whose coordinates satisfy the function. Thus, we graph $y = \sin x$ by listing some points on the graph. Because the period of the sine function is 2π, we will graph the function on the interval $[0, 2\pi]$. The rest of the graph is made up of repetitions of this portion.

Table 4.3 lists some values of (x, y) on the graph of $y = \sin x$, $0 \le x \le 2\pi$.

Table 4.3 Values of (x, y) on the graph of $y = \sin x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = \sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

As x increases from 0 to $\frac{\pi}{2}$, y increases from 0 to 1.

As x increases from $\frac{\pi}{2}$ to π, y decreases from 1 to 0.

As x increases from π to $\frac{3\pi}{2}$, y decreases from 0 to -1.

As x increases from $\frac{3\pi}{2}$ to 2π, y increases from -1 to 0.

In plotting the points obtained in **Table 4.3**, we will use the approximation $\frac{\sqrt{3}}{2} \approx 0.87$. Rather than approximating π, we will mark off units on the x-axis in terms of π. If we connect these points with a smooth curve, we obtain the graph shown in **Figure 4.62**. The figure shows one period of the graph of $y = \sin x$.

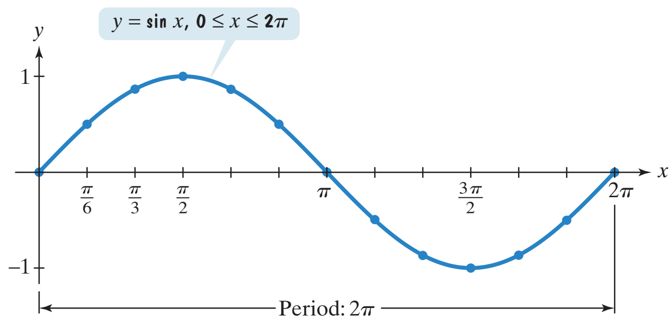


Figure 4.62 One period of the graph of $y = \sin x$

We can obtain a more complete graph of $y = \sin x$ by continuing the portion shown in **Figure 4.62** to the left and to the right. The graph of the sine function, called a **sine curve**, is shown in **Figure 4.63**. Any part of the graph that corresponds to one period (2π) is one cycle of the graph of $y = \sin x$.

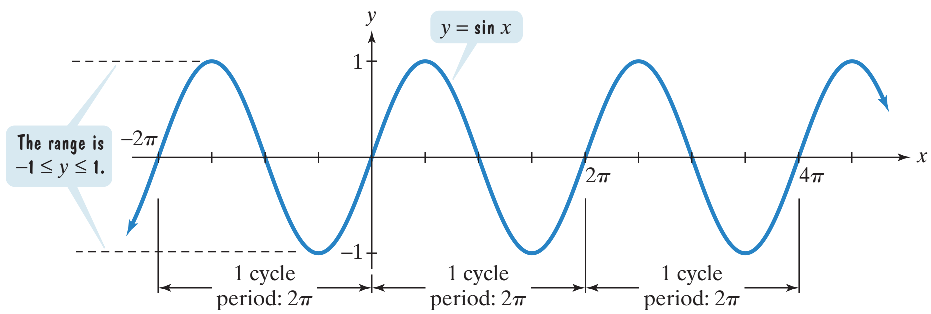


Figure 4.63 The graph of $y = \sin x$

The graph of $y = \sin x$ allows us to visualize some of the properties of the sine function.

- The domain is $(-\infty, \infty)$, the set of all real numbers. The graph extends indefinitely to the left and to the right with no gaps or holes.
- The range is $[-1, 1]$, the set of all real numbers between -1 and 1, inclusive. The graph never rises above 1 or falls below -1.
- The period is 2π. The graph's pattern repeats in every interval of length 2π.
- The function is an odd function: $\sin(-x) = -\sin x$. This can be seen by observing that the graph is symmetric with respect to the origin.

2 Graph variations of $y = \sin x$.

Graphing Variations of $y = \sin x$

To graph variations of $y = \sin x$ by hand, it is helpful to find x-intercepts, maximum points, and minimum points. One complete cycle of the sine curve includes three x-intercepts, one maximum point, and one minimum point. The graph of $y = \sin x$ has x-intercepts at the beginning, middle, and end of its full period, shown in **Figure 4.64**. The curve reaches its maximum point $\frac{1}{4}$ of the way through the period. It reaches its minimum point $\frac{3}{4}$ of the way through the period.

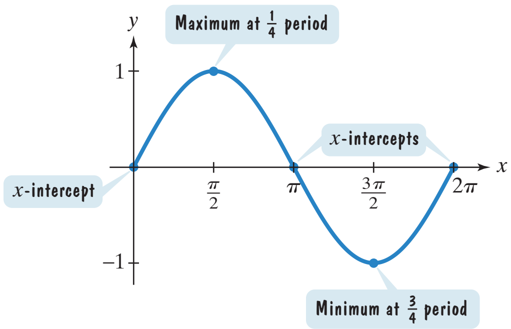


Figure 4.64 Key points in graphing the sine function

Thus, key points in graphing sine functions are obtained by dividing the period into four equal parts. The x-coordinates of the five key points are as follows:

$$x_1 = \text{value of } x \text{ where the cycle begins}$$

$$x_2 = x_1 + \frac{\text{period}}{4}$$

$$x_3 = x_2 + \frac{\text{period}}{4}$$

$$x_4 = x_3 + \frac{\text{period}}{4}$$

$$x_5 = x_4 + \frac{\text{period}}{4}.$$

Add "quarter-periods" to find successive values of x.

The y-coordinates of the five key points are obtained by evaluating the given function at each of these values of x.

The graph of $y = \sin x$ forms the basis for graphing functions of the form

$$y = A \sin x.$$

For example, consider $y = 2 \sin x$, in which $A = 2$. We can obtain the graph of $y = 2 \sin x$ from that of $y = \sin x$ if we multiply each y-coordinate on the graph of $y = \sin x$ by 2. **Figure 4.65** shows the graphs. The basic sine curve is *stretched* and ranges between -2 and 2, rather than between -1 and 1. However, both $y = \sin x$ and $y = 2 \sin x$ have a period of 2π.

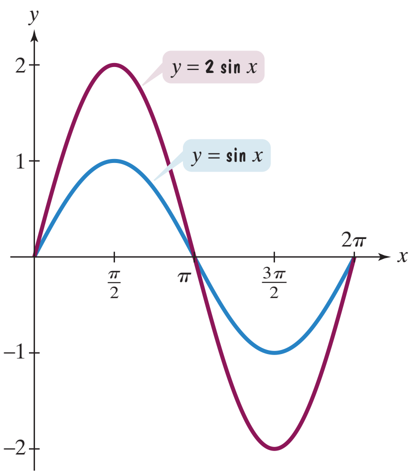


Figure 4.65 Comparing the graphs of $y = \sin x$ and $y = 2 \sin x$

In general, the graph of $y = A \sin x$ ranges between $-|A|$ and $|A|$. Thus, the range of the function is $-|A| \le y \le |A|$. If $|A| > 1$, the basic sine curve is *stretched*, as in **Figure 4.65**. If $|A| < 1$, the basic sine curve is *shrunk*. We call $|A|$ the **amplitude** of $y = A \sin x$. The maximum value of y on the graph of $y = A \sin x$ is $|A|$, the amplitude.

Graphing Variations of $y = \sin x$

1. Identify the amplitude and the period.
2. Find the values of x for the five key points—the three x-intercepts, the maximum point, and the minimum point. Start with the value of x where the cycle begins and add quarter-periods—that is, $\frac{\text{period}}{4}$—to find successive values of x.
3. Find the values of y for the five key points by evaluating the function at each value of x from step 2.
4. Connect the five key points with a smooth curve and graph one complete cycle of the given function.
5. Extend the graph in step 4 to the left or right as desired.

EXAMPLE 1 Graphing a Variation of $y = \sin x$

Determine the amplitude of $y = \frac{1}{2} \sin x$. Then graph $y = \sin x$ and $y = \frac{1}{2} \sin x$ for $0 \le x \le 2\pi$.

Solution

Step 1 Identify the amplitude and the period. The equation $y = \frac{1}{2} \sin x$ is of the form $y = A \sin x$ with $A = \frac{1}{2}$. Thus, the amplitude is $|A| = \frac{1}{2}$. This means that the maximum value of y is $\frac{1}{2}$ and the minimum value of y is $-\frac{1}{2}$. The period for both $y = \frac{1}{2} \sin x$ and $y = \sin x$ is 2π.

Step 2 Find the values of x for the five key points. We need to find the three x-intercepts, the maximum point, and the minimum point on the interval $[0, 2\pi]$. To do so, we begin by dividing the period, 2π, by 4.

$$\frac{\text{period}}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$$

We start with the value of x where the cycle begins: $x_1 = 0$. Now we add quarter-periods, $\frac{\pi}{2}$, to generate x-values for each of the key points. The five x-values are

$$x_1 = 0, \quad x_2 = 0 + \frac{\pi}{2} = \frac{\pi}{2}, \quad x_3 = \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$

$$x_4 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}, \quad x_5 = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi.$$

Step 3 Find the values of y for the five key points. We evaluate the function at each value of x from step 2.

Value of x	Value of y: $y = \frac{1}{2}\sin x$	Coordinates of key point
0	$y = \frac{1}{2}\sin 0 = \frac{1}{2} \cdot 0 = 0$	$(0, 0)$
$\frac{\pi}{2}$	$y = \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2}$	$\left(\frac{\pi}{2}, \frac{1}{2}\right)$
π	$y = \frac{1}{2}\sin \pi = \frac{1}{2} \cdot 0 = 0$	$(\pi, 0)$
$\frac{3\pi}{2}$	$y = \frac{1}{2}\sin\frac{3\pi}{2} = \frac{1}{2}(-1) = -\frac{1}{2}$	$\left(\frac{3\pi}{2}, -\frac{1}{2}\right)$
2π	$y = \frac{1}{2}\sin 2\pi = \frac{1}{2} \cdot 0 = 0$	$(2\pi, 0)$

maximum point

minimum point

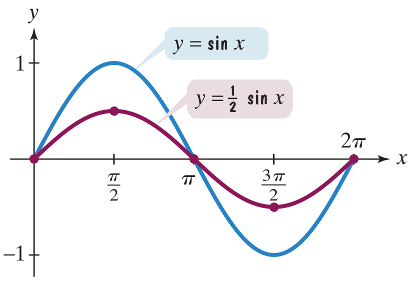


Figure 4.66 The graphs of $y = \sin x$ and $y = \frac{1}{2}\sin x$, $0 \le x \le 2\pi$

There are x-intercepts at 0, π, and 2π. The maximum and minimum points are indicated by the voice balloons.

Step 4 Connect the five key points with a smooth curve and graph one complete cycle of the given function. The five key points for $y = \frac{1}{2}\sin x$ are shown in **Figure 4.66**. By connecting the points with a smooth curve, the figure shows one complete cycle of $y = \frac{1}{2}\sin x$. Also shown is the graph of $y = \sin x$. The graph of $y = \frac{1}{2}\sin x$ is the graph of $y = \sin x$ vertically shrunk by a factor of $\frac{1}{2}$.

Check Point 1 Determine the amplitude of $y = 3\sin x$. Then graph $y = \sin x$ and $y = 3\sin x$ for $0 \le x \le 2\pi$.

EXAMPLE 2 Graphing a Variation of $y = \sin x$

Determine the amplitude of $y = -2\sin x$. Then graph $y = \sin x$ and $y = -2\sin x$ for $-\pi \le x \le 3\pi$.

Solution

Step 1 Identify the amplitude and the period. The equation $y = -2\sin x$ is of the form $y = A\sin x$ with $A = -2$. Thus, the amplitude is $|A| = |-2| = 2$. This means that the maximum value of y is 2 and the minimum value of y is -2. Both $y = \sin x$ and $y = -2\sin x$ have a period of 2π.

Step 2 Find the x-values for the five key points. Begin by dividing the period, 2π, by 4.

$$\frac{\text{period}}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Start with the value of x where the cycle begins: $x_1 = 0$. Adding quarter-periods, $\frac{\pi}{2}$, the five x-values for the key points are

$$x_1 = 0, \quad x_2 = 0 + \frac{\pi}{2} = \frac{\pi}{2}, \quad x_3 = \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$

$$x_4 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}, \quad x_5 = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi.$$

Although we will be graphing on $[-\pi, 3\pi]$, we select $x_1 = 0$ rather than $x_1 = -\pi$. Knowing the graph's shape on $[0, 2\pi]$ will enable us to continue the pattern and extend it to the left to $-\pi$ and to the right to 3π.

Step 3 Find the values of y for the five key points. We evaluate the function at each value of x from step 2.

Value of x	Value of y: $y = -2 \sin x$	Coordinates of key point
0	$y = -2 \sin 0 = -2 \cdot 0 = 0$	$(0, 0)$
$\frac{\pi}{2}$	$y = -2 \sin \frac{\pi}{2} = -2 \cdot 1 = -2$	$\left(\frac{\pi}{2}, -2\right)$ minimum point
π	$y = -2 \sin \pi = -2 \cdot 0 = 0$	$(\pi, 0)$
$\frac{3\pi}{2}$	$y = -2 \sin \frac{3\pi}{2} = -2(-1) = 2$	$\left(\frac{3\pi}{2}, 2\right)$ maximum point
2π	$y = -2 \sin 2\pi = -2 \cdot 0 = 0$	$(2\pi, 0)$

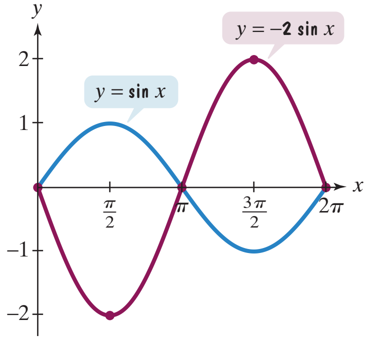


Figure 4.67 The graphs of $y = \sin x$ and $y = -2 \sin x$, $0 \le x \le 2\pi$

There are x-intercepts at 0, π, and 2π. The minimum and maximum points are indicated by the voice balloons.

Step 4 Connect the five key points with a smooth curve and graph one complete cycle of the given function. The five key points for $y = -2 \sin x$ are shown in **Figure 4.67**. By connecting the points with a smooth curve, the dark red portion shows one complete cycle of $y = -2 \sin x$. Also shown in dark blue is one complete cycle of the graph of $y = \sin x$. The graph of $y = -2 \sin x$ is the graph of $y = \sin x$ reflected about the x-axis and vertically stretched by a factor of 2.

Step 5 Extend the graph in step 4 to the left or right as desired. The dark red and dark blue portions of the graphs in **Figure 4.67** are from 0 to 2π. In order to graph for $-\pi \le x \le 3\pi$, continue the pattern of each graph to the left and to the right. These extensions are shown by the lighter colors in **Figure 4.68**.

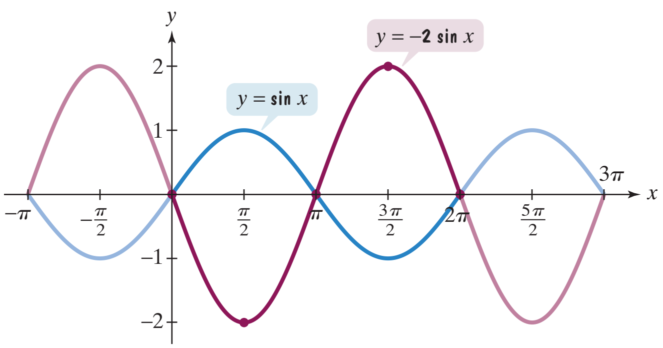


Figure 4.68 The graphs of $y = \sin x$ and $y = -2 \sin x$, $-\pi \le x \le 3\pi$

Check Point 2 Determine the amplitude of $y = -\frac{1}{2} \sin x$. Then graph $y = \sin x$ and $y = -\frac{1}{2} \sin x$ for $-\pi \le x \le 3\pi$.

Study Tip

If $B < 0$ in $y = A \sin Bx$, use $\sin(-\theta) = -\sin\theta$ to rewrite the equation before obtaining its graph.

Now let us examine the graphs of functions of the form $y = A \sin Bx$, where B is the coefficient of x and $B > 0$. How do such graphs compare to those of functions of the form $y = A \sin x$? We know that $y = A \sin x$ completes one cycle from $x = 0$ to $x = 2\pi$. Thus, $y = A \sin Bx$ completes one cycle as Bx increases from 0 to 2π. Set up an inequality to represent this and solve for x to determine the values of x for which $y = \sin Bx$ completes one cycle.

$$0 \le Bx \le 2\pi \qquad y = \sin Bx \text{ completes one cycle as } Bx \text{ increases from 0 to } 2\pi.$$

$$0 \le x \le \frac{2\pi}{B} \qquad \text{Divide by } B\text{, where } B > 0\text{, and solve for } x.$$

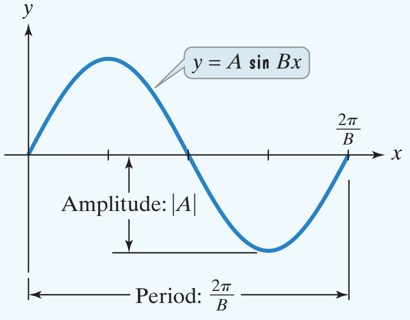
The inequality $0 \le x \le \frac{2\pi}{B}$ means that $y = A \sin Bx$ completes one cycle from 0 to $\frac{2\pi}{B}$. The period is $\frac{2\pi}{B}$. The graph of $y = A \sin Bx$ is the graph of $y = A \sin x$ horizontally shrunk by a factor of $\frac{1}{B}$ if $B > 1$ and horizontally stretched by a factor of $\frac{1}{B}$ if $0 < B < 1$.

Amplitudes and Periods

The graph of $y = A \sin Bx$ has

$$\text{amplitude} = |A|$$

$$\text{period} = \frac{2\pi}{B}.$$



EXAMPLE 3 Graphing a Function of the Form $y = A \sin Bx$

Determine the amplitude and period of $y = 3 \sin 2x$. Then graph the function for $0 \le x \le 2\pi$.

Solution

Step 1 Identify the amplitude and the period. The equation $y = 3 \sin 2x$ is of the form $y = A \sin Bx$ with $A = 3$ and $B = 2$.

$$\text{amplitude:} \quad |A| = |3| = 3$$

$$\text{period:} \quad \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

The amplitude, 3, tells us that the maximum value of y is 3 and the minimum value of y is -3. The period, π, tells us that the graph completes one cycle from 0 to π.

Step 2 Find the x-values for the five key points. Begin by dividing the period of $y = 3 \sin 2x$, π, by 4.

$$\frac{\text{period}}{4} = \frac{\pi}{4}$$

Start with the value of x where the cycle begins: $x_1 = 0$. Adding quarter-periods, $\frac{\pi}{4}$, the five x-values for the key points are

$$x_1 = 0, \quad x_2 = 0 + \frac{\pi}{4} = \frac{\pi}{4}, \quad x_3 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2},$$

$$x_4 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}, \quad x_5 = \frac{3\pi}{4} + \frac{\pi}{4} = \pi.$$

Step 3 Find the values of y for the five key points. We evaluate the function at each value of x from step 2.

Value of x	Value of y: $y = 3 \sin 2x$	Coordinates of key point
0	$y = 3 \sin (2 \cdot 0)$ $= 3 \sin 0 = 3 \cdot 0 = 0$	$(0, 0)$
$\frac{\pi}{4}$	$y = 3 \sin\left(2 \cdot \frac{\pi}{4}\right)$ $= 3 \sin \frac{\pi}{2} = 3 \cdot 1 = 3$	$\left(\frac{\pi}{4}, 3\right)$ maximum point
$\frac{\pi}{2}$	$y = 3 \sin\left(2 \cdot \frac{\pi}{2}\right)$ $= 3 \sin \pi = 3 \cdot 0 = 0$	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$y = 3 \sin\left(2 \cdot \frac{3\pi}{4}\right)$ $= 3 \sin \frac{3\pi}{2} = 3(-1) = -3$	$\left(\frac{3\pi}{4}, -3\right)$ minimum point
π	$y = 3 \sin (2 \cdot \pi)$ $= 3 \sin 2\pi = 3 \cdot 0 = 0$	$(\pi, 0)$

In the interval $[0, \pi]$, there are x-intercepts at $0, \frac{\pi}{2}$, and π. The maximum and minimum points are indicated by the voice balloons.

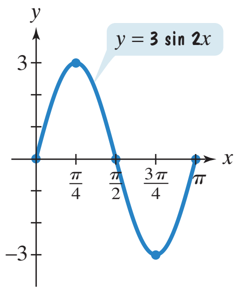


Figure 4.69 The graph of $y = 3 \sin 2x, 0 \le x \le \pi$

Step 4 Connect the five key points with a smooth curve and graph one complete cycle of the given function. The five key points for $y = 3 \sin 2x$ are shown in **Figure 4.69**. By connecting the points with a smooth curve, the blue portion shows one complete cycle of $y = 3 \sin 2x$ from 0 to π. The graph of $y = 3 \sin 2x$ is the graph of $y = \sin x$ vertically stretched by a factor of 3 and horizontally shrunk by a factor of $\frac{1}{2}$.

Step 5 Extend the graph in step 4 to the left or right as desired. The blue portion of the graph in **Figure 4.69** is from 0 to π. In order to graph for $0 \le x \le 2\pi$, we continue this portion and extend the graph another full period to the right. This extension is shown in gray in **Figure 4.70**.

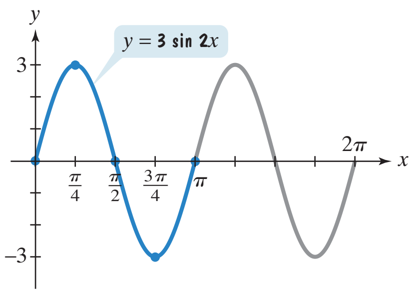


Figure 4.70

Check Point 3 Determine the amplitude and period of $y = 2 \sin \frac{1}{2}x$. Then graph the function for $0 \le x \le 8\pi$.

Now let us examine the graphs of functions of the form $y = A \sin(Bx - C)$, where $B > 0$. How do such graphs compare to those of functions of the form $y = A \sin Bx$? In both cases, the amplitude is $|A|$ and the period is $\frac{2\pi}{B}$. One complete cycle occurs if $Bx - C$ increases from 0 to 2π. This means that we can find an interval containing one cycle by solving the following inequality:

$$0 \le Bx - C \le 2\pi.$$

$y = A \sin(Bx - C)$ completes one cycle as $Bx - C$ increases from 0 to 2π.

$$C \le Bx \le C + 2\pi$$

Add C to all three parts.

$$\frac{C}{B} \le x \le \frac{C}{B} + \frac{2\pi}{B}$$

Divide by B, where $B > 0$, and solve for x.

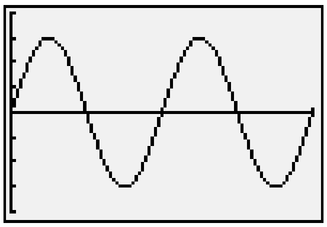
This is the x-coordinate on the left where the cycle begins.

This is the x-coordinate on the right where the cycle ends. $\frac{2\pi}{B}$ is the period.

The voice balloon on the left indicates that the graph of $y = A \sin(Bx - C)$ is the graph of $y = A \sin Bx$ shifted horizontally by $\frac{C}{B}$. Thus, the number $\frac{C}{B}$ is the **phase shift** associated with the graph.

Technology

The graph of $y = 3 \sin 2x$ in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-4, 4, 1]$ viewing rectangle verifies our hand-drawn graph in **Figure 4.70**.

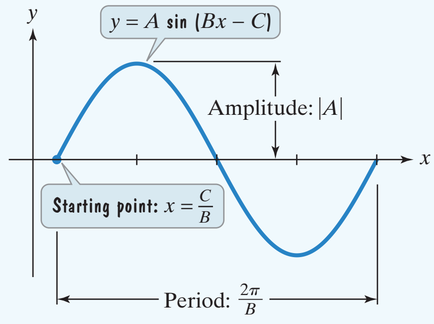


The Graph of $y = A\sin(Bx - C)$

The graph of $y = A\sin(Bx - C)$ is obtained by horizontally shifting the graph of $y = A\sin Bx$ so that the starting point of the cycle is shifted from $x = 0$ to $x = \frac{C}{B}$. If $\frac{C}{B} > 0$, the shift is to the right. If $\frac{C}{B} < 0$, the shift is to the left. The number $\frac{C}{B}$ is called the **phase shift**.

$$\text{amplitude} = |A|$$

$$\text{period} = \frac{2\pi}{B}$$



EXAMPLE 4 Graphing a Function of the Form $y = A\sin(Bx - C)$

Determine the amplitude, period, and phase shift of $y = 4\sin\left(2x - \frac{2\pi}{3}\right)$. Then graph one period of the function.

Solution

Step 1 Identify the amplitude, the period, and the phase shift. We must first identify values for A, B, and C.

The equation is of the form $y = A\sin(Bx - C)$.

$$y = 4\sin\left(2x - \frac{2\pi}{3}\right)$$

Using the voice balloon, we see that $A = 4$, $B = 2$, and $C = \frac{2\pi}{3}$.

$$\text{amplitude: } |A| = |4| = 4$$

The maximum y is 4 and the minimum is -4.

$$\text{period: } \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

Each cycle is of length π.

$$\text{phase shift: } \frac{C}{B} = \frac{\frac{2\pi}{3}}{2} = \frac{2\pi}{3}\cdot\frac{1}{2} = \frac{\pi}{3}$$

A cycle starts at $x = \frac{\pi}{3}$.

Step 2 Find the x-values for the five key points. Begin by dividing the period, π, by 4.

$$\frac{\text{period}}{4} = \frac{\pi}{4}$$

Start with the value of x where the cycle begins: $x_1 = \frac{\pi}{3}$. Then add quarter-periods, $\frac{\pi}{4}$.

Study Tip

You can speed up the additions on the right by first writing the starting point, $\frac{\pi}{3}$, and the quarter-period, $\frac{\pi}{4}$, with a common denominator, 12.

$$\text{starting point} = \frac{\pi}{3} = \frac{4\pi}{12}$$

$$\text{quarter-period} = \frac{\pi}{4} = \frac{3\pi}{12}$$

Starting with where the cycle begins $\left(x_1 = \frac{\pi}{3}\right)$ and adding quarter periods, $\frac{\pi}{4}$, the five x-values of the key points are

$$x_1 = \frac{\pi}{3}, \quad x_2 = \frac{\pi}{3} + \frac{\pi}{4} = \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{7\pi}{12},$$

$$x_3 = \frac{7\pi}{12} + \frac{\pi}{4} = \frac{7\pi}{12} + \frac{3\pi}{12} = \frac{10\pi}{12} = \frac{5\pi}{6},$$

$$x_4 = \frac{5\pi}{6} + \frac{\pi}{4} = \frac{10\pi}{12} + \frac{3\pi}{12} = \frac{13\pi}{12},$$

$$x_5 = \frac{13\pi}{12} + \frac{\pi}{4} = \frac{13\pi}{12} + \frac{3\pi}{12} = \frac{16\pi}{12} = \frac{4\pi}{3}.$$

Study Tip

You can check your computations for the x-values for the five key points. The difference between x_5 and x_1, or $x_5 - x_1$, should equal the period.

$$x_5 - x_1 = \frac{4\pi}{3} - \frac{\pi}{3} = \frac{3\pi}{3} = \pi$$

Because the period is π, this verifies that our five x-values are correct.

Step 3 Find the values of y for the five key points. We evaluate the function at each value of x from step 2.

Value of x	Value of y: $y = 4\sin\left(2x - \frac{2\pi}{3}\right)$	Coordinates of key point
$\frac{\pi}{3}$	$y = 4\sin\left(2\cdot\frac{\pi}{3} - \frac{2\pi}{3}\right)$ $= 4\sin 0 = 4\cdot 0 = 0$	$\left(\frac{\pi}{3}, 0\right)$
$\frac{7\pi}{12}$	$y = 4\sin\left(2\cdot\frac{7\pi}{12} - \frac{2\pi}{3}\right)$ $= 4\sin\left(\frac{7\pi}{6} - \frac{2\pi}{3}\right)$ $= 4\sin\frac{3\pi}{6} = 4\sin\frac{\pi}{2} = 4\cdot 1 = 4$	$\left(\frac{7\pi}{12}, 4\right)$ maximum point
$\frac{5\pi}{6}$	$y = 4\sin\left(2\cdot\frac{5\pi}{6} - \frac{2\pi}{3}\right)$ $= 4\sin\left(\frac{5\pi}{3} - \frac{2\pi}{3}\right)$ $= 4\sin\frac{3\pi}{3} = 4\sin\pi = 4\cdot 0 = 0$	$\left(\frac{5\pi}{6}, 0\right)$
$\frac{13\pi}{12}$	$y = 4\sin\left(2\cdot\frac{13\pi}{12} - \frac{2\pi}{3}\right)$ $= 4\sin\left(\frac{13\pi}{6} - \frac{4\pi}{6}\right)$ $= 4\sin\frac{9\pi}{6} = 4\sin\frac{3\pi}{2} = 4(-1) = -4$	$\left(\frac{13\pi}{12}, -4\right)$ minimum point
$\frac{4\pi}{3}$	$y = 4\sin\left(2\cdot\frac{4\pi}{3} - \frac{2\pi}{3}\right)$ $= 4\sin\frac{6\pi}{3} = 4\sin 2\pi = 4\cdot 0 = 0$	$\left(\frac{4\pi}{3}, 0\right)$

In the interval $\left[\frac{\pi}{3}, \frac{4\pi}{3}\right]$, there are x-intercepts at $\frac{\pi}{3}$, $\frac{5\pi}{6}$, and $\frac{4\pi}{3}$. The maximum and minimum points are indicated by the voice balloons.

Step 4 Connect the five key points with a smooth curve and graph one complete cycle of the given function. The five key points are shown on the graph of $y = 4\sin\left(2x - \frac{2\pi}{3}\right)$ in **Figure 4.71**.

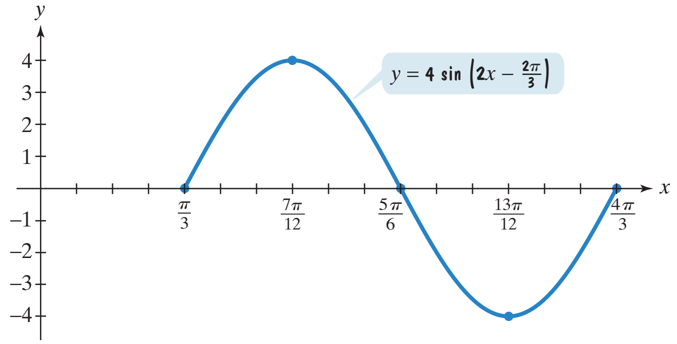


Figure 4.71

Check Point 4 Determine the amplitude, period, and phase shift of $y = 3\sin\left(2x - \frac{\pi}{3}\right)$. Then graph one period of the function.

3 Understand the graph of $y = \cos x$.

The Graph of $y = \cos x$

We graph $y = \cos x$ by listing some points on the graph. Because the period of the cosine function is 2π, we will concentrate on the graph of the basic cosine curve on the interval $[0, 2\pi]$. The rest of the graph is made up of repetitions of this portion. **Table 4.4** lists some values of (x, y) on the graph of $y = \cos x$.

Table 4.4 Values of (x, y) on the Graph of $y = \cos x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = \cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

As x increases from 0 to $\frac{\pi}{2}$, y decreases from 1 to 0.

As x increases from $\frac{\pi}{2}$ to π, y decreases from 0 to −1.

As x increases from π to $\frac{3\pi}{2}$, y increases from −1 to 0.

As x increases from $\frac{3\pi}{2}$ to 2π, y increases from 0 to 1.

Plotting the points in **Table 4.4** and connecting them with a smooth curve, we obtain the graph shown in **Figure 4.72**. The portion of the graph in dark blue shows one complete period. We can obtain a more complete graph of $y = \cos x$ by extending this dark blue portion to the left and to the right.

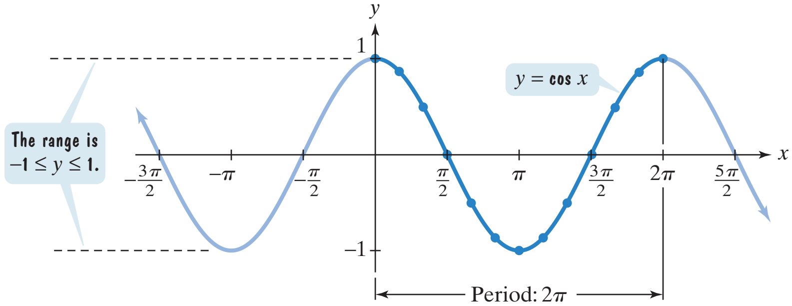


Figure 4.72 The graph of $y = \cos x$

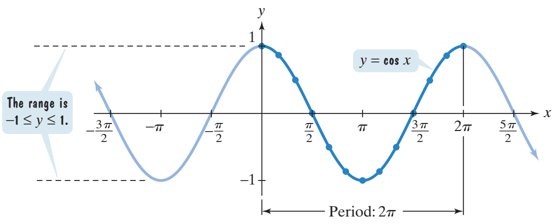


Figure 4.72 The graph of $y = \cos x$ (repeated)

The graph of $y = \cos x$ allows us to visualize some of the properties of the cosine function.

- The domain is $(-\infty, \infty)$, the set of all real numbers. The graph extends indefinitely to the left and to the right with no gaps or holes.
- The range is $[-1, 1]$, the set of all real numbers between -1 and 1, inclusive. The graph never rises above 1 or falls below -1.
- The period is 2π. The graph's pattern repeats in every interval of length 2π.
- The function is an even function: $\cos(-x) = \cos x$. This can be seen by observing that the graph is symmetric with respect to the y-axis.

Take a second look at **Figure 4.72**. Can you see that the graph of $y = \cos x$ is the graph of $y = \sin x$ with a phase shift of $-\frac{\pi}{2}$? If you trace along the curve from $x = -\frac{\pi}{2}$ to $x = \frac{3\pi}{2}$, you are tracing one complete cycle of the sine curve. This can be expressed as an identity:

$$\cos x = \sin\left(x + \frac{\pi}{2}\right).$$

Because of this similarity, the graphs of sine functions and cosine functions are called **sinusoidal graphs**.

4 Graph variations of $y = \cos x$.

Graphing Variations of $y = \cos x$

We use the same steps to graph variations of $y = \cos x$ as we did for graphing variations of $y = \sin x$. We will continue finding key points by dividing the period into four equal parts. Amplitudes, periods, and phase shifts play an important role when graphing by hand.

Study Tip

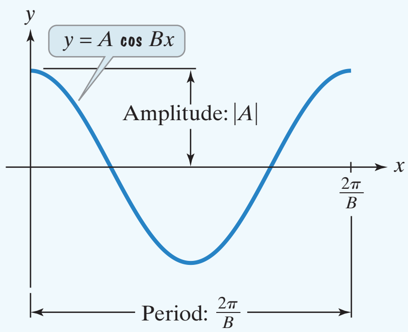
If $B < 0$ in $y = A \cos Bx$, use $\cos(-\theta) = \cos\theta$ to rewrite the equation before obtaining its graph.

The Graph of $y = A \cos Bx$

The graph of $y = A \cos Bx$ has

$$\text{amplitude} = |A|$$

$$\text{period} = \frac{2\pi}{B}.$$



EXAMPLE 5 Graphing a Function of the Form $y = A \cos Bx$

Determine the amplitude and period of $y = -3 \cos \frac{\pi}{2}x$. Then graph the function for $-4 \le x \le 4$.

Solution

Step 1 Identify the amplitude and the period. The equation $y = -3 \cos \frac{\pi}{2}x$ is of the form $y = A \cos Bx$ with $A = -3$ and $B = \frac{\pi}{2}$.

$$\text{amplitude: } |A| = |-3| = 3$$

The maximum y is 3 and the minimum is -3.

$$\text{period: } \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{2}} = 2\cancel{\pi} \cdot \frac{2}{\cancel{\pi}} = 4$$

Each cycle is of length 4.

Step 2 Find the x-values for the five key points. Begin by dividing the period, 4, by 4.

$$\frac{\text{period}}{4} = \frac{4}{4} = 1$$

Start with the value of x where the cycle begins: $x_1 = 0$. Adding quarter-periods, 1, the five x-values for the key points are

$$x_1 = 0, \quad x_2 = 0 + 1 = 1, \quad x_3 = 1 + 1 = 2, \quad x_4 = 2 + 1 = 3, \quad x_5 = 3 + 1 = 4.$$

Step 3 Find the values of y for the five key points. We evaluate the function at each value of x from step 2.

Value of x	Value of y: $y = -3\cos\frac{\pi}{2}x$	Coordinates of key point
0	$y = -3\cos\left(\frac{\pi}{2}\cdot 0\right)$ $= -3\cos 0 = -3\cdot 1 = -3$	$(0, -3)$
1	$y = -3\cos\left(\frac{\pi}{2}\cdot 1\right)$ $= -3\cos\frac{\pi}{2} = -3\cdot 0 = 0$	$(1, 0)$
2	$y = -3\cos\left(\frac{\pi}{2}\cdot 2\right)$ $= -3\cos\pi = -3(-1) = 3$	$(2, 3)$
3	$y = -3\cos\left(\frac{\pi}{2}\cdot 3\right)$ $= -3\cos\frac{3\pi}{2} = -3\cdot 0 = 0$	$(3, 0)$
4	$y = -3\cos\left(\frac{\pi}{2}\cdot 4\right)$ $= -3\cos 2\pi = -3\cdot 1 = -3$	$(4, -3)$

minimum point

maximum point

minimum point

In the interval $[0, 4]$, there are x-intercepts at 1 and 3. The minimum and maximum points are indicated by the voice balloons.

Step 4 Connect the five key points with a smooth curve and graph one complete cycle of the given function. The five key points for $y = -3\cos\frac{\pi}{2}x$ are shown in **Figure 4.73**. By connecting the points with a smooth curve, the blue portion shows one complete cycle of $y = -3\cos\frac{\pi}{2}x$ from 0 to 4.

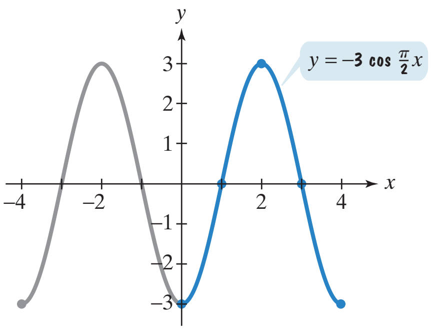
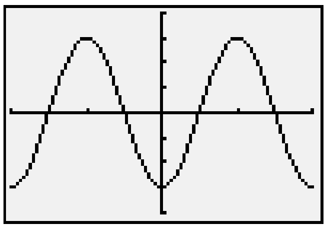


Figure 4.73

Step 5 Extend the graph in step 4 to the left or right as desired. The blue portion of the graph in **Figure 4.73** is for x from 0 to 4. In order to graph for $-4 \le x \le 4$, we continue this portion and extend the graph another full period to the left. This extension is shown in gray in **Figure 4.73**.

Technology

The graph of $y = -3\cos\frac{\pi}{2}x$ in a $[-4, 4, 1]$ by $[-4, 4, 1]$ viewing rectangle verifies our hand-drawn graph in **Figure 4.73**.



Check Point 5 Determine the amplitude and period of $y = -4\cos\pi x$. Then graph the function for $-2 \le x \le 2$.

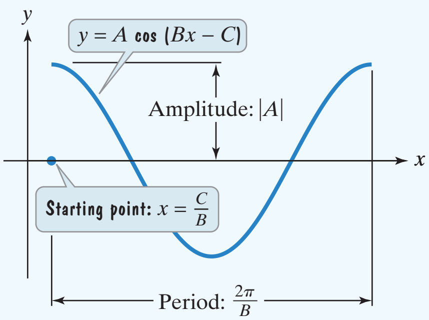
Finally, let us examine the graphs of functions of the form $y = A\cos(Bx - C)$. Graphs of these functions shift the graph of $y = A\cos Bx$ horizontally by $\frac{C}{B}$.

The Graph of $y = A\cos(Bx - C)$

The graph of $y = A\cos(Bx - C)$ is obtained by horizontally shifting the graph of $y = A\cos Bx$ so that the starting point of the cycle is shifted from $x = 0$ to $x = \frac{C}{B}$. If $\frac{C}{B} > 0$, the shift is to the right. If $\frac{C}{B} < 0$, the shift is to the left. The number $\frac{C}{B}$ is called the **phase shift**.

$$\text{amplitude} = |A|$$

$$\text{period} = \frac{2\pi}{B}$$



EXAMPLE 6 Graphing a Function of the Form $y = A\cos(Bx - C)$

Determine the amplitude, period, and phase shift of $y = \frac{1}{2}\cos(4x + \pi)$. Then graph one period of the function.

Solution

Step 1 Identify the amplitude, the period, and the phase shift. We must first identify values for A, B, and C. To do this, we need to express the equation in the form $y = A\cos(Bx - C)$. Thus, we write $y = \frac{1}{2}\cos(4x + \pi)$ as $y = \frac{1}{2}\cos[4x - (-\pi)]$. Now we can identify values for A, B, and C.

The equation is of the form $y = A\cos(Bx - C)$.

$$y = \frac{1}{2}\cos[4x - (-\pi)]$$

Using the voice balloon, we see that $A = \frac{1}{2}$, $B = 4$, and $C = -\pi$.

$$\text{amplitude: } |A| = \left|\frac{1}{2}\right| = \frac{1}{2}$$

The maximum y is $\frac{1}{2}$ and the minimum is $-\frac{1}{2}$.

$$\text{period: } \frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Each cycle is of length $\frac{\pi}{2}$.

$$\text{phase shift: } \frac{C}{B} = -\frac{\pi}{4}$$

A cycle starts at $x = -\frac{\pi}{4}$.

Step 2 Find the x-values for the five key points. Begin by dividing the period, $\frac{\pi}{2}$, by 4.

$$\frac{\text{period}}{4} = \frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$$

Start with the value of x where the cycle begins: $x_1 = -\frac{\pi}{4}$. Adding quarter-periods, $\frac{\pi}{8}$, the five x-values for the key points are

$$x_1 = -\frac{\pi}{4}, \quad x_2 = -\frac{\pi}{4} + \frac{\pi}{8} = -\frac{2\pi}{8} + \frac{\pi}{8} = -\frac{\pi}{8}, \quad x_3 = -\frac{\pi}{8} + \frac{\pi}{8} = 0,$$

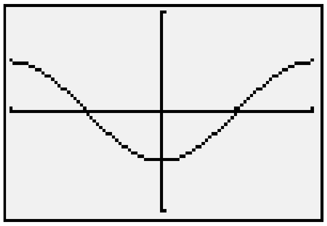
$$x_4 = 0 + \frac{\pi}{8} = \frac{\pi}{8}, \quad x_5 = \frac{\pi}{8} + \frac{\pi}{8} = \frac{2\pi}{8} = \frac{\pi}{4}.$$

Technology

The graph of

$$y = \frac{1}{2}\cos(4x + \pi)$$

in a $\left[-\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{8}\right]$ by $[-1, 1, 1]$ viewing rectangle verifies our hand-drawn graph in **Figure 4.74**.



Step 3 Find the values of y for the five key points. Take a few minutes and use your calculator to evaluate the function at each value of x from step 2. Show that the key points are

$$\left(-\frac{\pi}{4}, \frac{1}{2}\right), \quad \left(-\frac{\pi}{8}, 0\right), \quad \left(0, -\frac{1}{2}\right), \quad \left(\frac{\pi}{8}, 0\right), \text{ and } \left(\frac{\pi}{4}, \frac{1}{2}\right).$$

maximum point; x-intercept at $-\frac{\pi}{8}$; minimum point; x-intercept at $\frac{\pi}{8}$; maximum point

Step 4 Connect the five key points with a smooth curve and graph one complete cycle of the given function. The key points and the graph of $y = \frac{1}{2}\cos(4x + \pi)$ are shown in **Figure 4.74**.

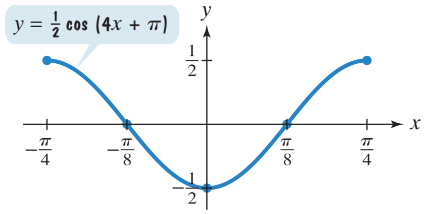


Figure 4.74

Check Point 6 Determine the amplitude, period, and phase shift of $y = \frac{3}{2}\cos(2x + \pi)$. Then graph one period of the function.

5 Use vertical shifts of sine and cosine curves.

Vertical Shifts of Sinusoidal Graphs

We now look at sinusoidal graphs of

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D.$$

The constant D causes vertical shifts in the graphs of $y = A\sin(Bx - C)$ and $y = A\cos(Bx - C)$. If D is positive, the shift is D units upward. If D is negative, the shift is $|D|$ units downward. These vertical shifts result in sinusoidal graphs oscillating about the horizontal line $y = D$ rather than about the x-axis. Thus, the maximum y is $D + |A|$ and the minimum y is $D - |A|$.

EXAMPLE 7 A Vertical Shift

Graph one period of the function $y = \frac{1}{2}\cos x - 1$.

Solution The graph of $y = \frac{1}{2}\cos x - 1$ is the graph of $y = \frac{1}{2}\cos x$ shifted one unit downward. The period of $y = \frac{1}{2}\cos x$ is 2π, which is also the period for the vertically shifted graph. The key points on the interval $[0, 2\pi]$ for $y = \frac{1}{2}\cos x - 1$ are found by first determining their x-coordinates. The quarter-period is $\frac{2\pi}{4}$, or $\frac{\pi}{2}$. The cycle begins at $x = 0$. As always, we add quarter-periods to generate x-values for each of the key points. The five x-values are

$$x_1 = 0, \quad x_2 = 0 + \frac{\pi}{2} = \frac{\pi}{2}, \quad x_3 = \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$

$$x_4 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}, \quad x_5 = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi.$$

The values of y for the five key points and their coordinates are determined as follows.

Value of x	Value of y: $y = \frac{1}{2}\cos x - 1$	Coordinates of Key Point
0	$y = \frac{1}{2}\cos 0 - 1$ $= \frac{1}{2} \cdot 1 - 1 = -\frac{1}{2}$	$\left(0, -\frac{1}{2}\right)$
$\frac{\pi}{2}$	$y = \frac{1}{2}\cos \frac{\pi}{2} - 1$ $= \frac{1}{2} \cdot 0 - 1 = -1$	$\left(\frac{\pi}{2}, -1\right)$
π	$y = \frac{1}{2}\cos \pi - 1$ $= \frac{1}{2}(-1) - 1 = -\frac{3}{2}$	$\left(\pi, -\frac{3}{2}\right)$
$\frac{3\pi}{2}$	$y = \frac{1}{2}\cos \frac{3\pi}{2} - 1$ $= \frac{1}{2} \cdot 0 - 1 = -1$	$\left(\frac{3\pi}{2}, -1\right)$
2π	$y = \frac{1}{2}\cos 2\pi - 1$ $= \frac{1}{2} \cdot 1 - 1 = -\frac{1}{2}$	$\left(2\pi, -\frac{1}{2}\right)$

The five key points for $y = \frac{1}{2}\cos x - 1$ are shown in **Figure 4.75**. By connecting the points with a smooth curve, we obtain one period of the graph.

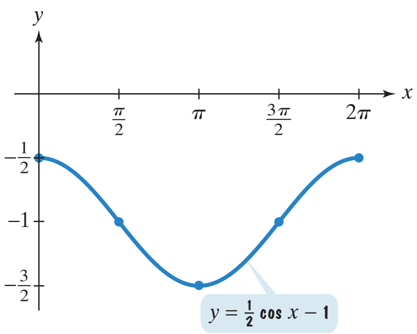


Figure 4.75

Check Point 7 Graph one period of the function $y = 2\cos x + 1$.

6 Model periodic behavior.

Modeling Periodic Behavior

Our breathing consists of alternating periods of inhaling and exhaling. Each complete pumping cycle of the human heart can be described using a sine function. Our brain waves during deep sleep are sinusoidal. Viewed in this way, trigonometry becomes an intimate experience.

Some graphing utilities have a SINe REGression feature. This feature gives the sine function in the form $y = A\sin(Bx + C) + D$ of best fit for wavelike data. At least four data points must be used. However, it is not always necessary to use technology. In our next example, we use our understanding of sinusoidal graphs to model the process of breathing.

EXAMPLE 8 A Trigonometric Breath of Life

The graph in **Figure 4.76** shows one complete normal breathing cycle. The cycle consists of inhaling and exhaling. It takes place every 5 seconds. Velocity of air flow is positive when we inhale and negative when we exhale. It is measured in liters per second. If y represents velocity of air flow after x seconds, find a function of the form $y = A \sin Bx$ that models air flow in a normal breathing cycle.

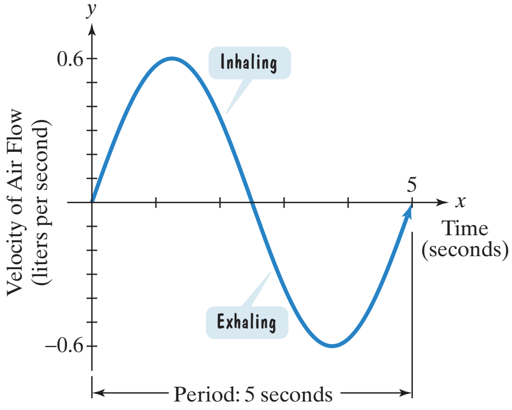


Figure 4.76

Solution We need to determine values for A and B in the equation $y = A \sin Bx$. The amplitude, A, is the maximum value of y. **Figure 4.76** shows that this maximum value is 0.6. Thus, $A = 0.6$.

The value of B in $y = A \sin Bx$ can be found using the formula for the period: period $= \frac{2\pi}{B}$. The period of our breathing cycle is 5 seconds. Thus,

$$5 = \frac{2\pi}{B} \quad \text{Our goal is to solve this equation for } B.$$

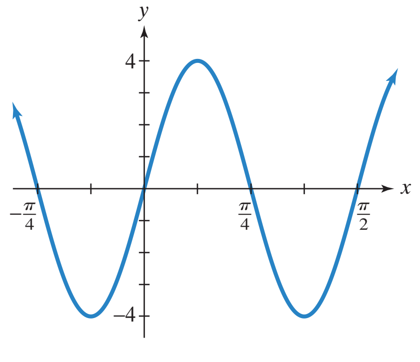
$$5B = 2\pi \quad \text{Multiply both sides of the equation by } B.$$

$$B = \frac{2\pi}{5}. \quad \text{Divide both sides of the equation by 5.}$$

We see that $A = 0.6$ and $B = \frac{2\pi}{5}$. Substitute these values into $y = A \sin Bx$. The breathing cycle is modeled by

$$y = 0.6 \sin \frac{2\pi}{5}x.$$

Check Point 8 Find an equation of the form $y = A \sin Bx$ that produces the graph shown in the figure on the right.



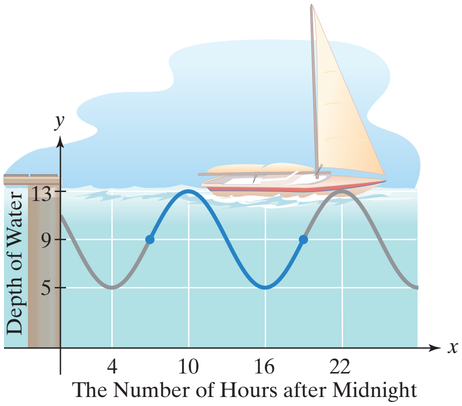


Figure 4.77

EXAMPLE 9 Modeling a Tidal Cycle

Figure 4.77 shows that the depth of water at a boat dock varies with the tides. The depth is 5 feet at low tide and 13 feet at high tide. On a certain day, low tide occurs at 4 A.M. and high tide at 10 A.M. If y represents the depth of the water, in feet, x hours after midnight, use a sine function of the form $y = A\sin(Bx - C) + D$ to model the water's depth.

Solution We need to determine values for $A, B, C,$ and D in the equation $y = A\sin(Bx - C) + D$. We can find these values using **Figure 4.77**. We begin with D.

To find D, we use the vertical shift. Because the water's depth ranges from a minimum of 5 feet to a maximum of 13 feet, the curve oscillates about the middle value, 9 feet. Thus, $D = 9$, which is the vertical shift.

At maximum depth, the water is 4 feet above 9 feet. Thus, A, the amplitude, is 4: $A = 4$.

To find B, we use the period. The blue portion of the graph shows that one complete tidal cycle occurs in $19 - 7$, or 12 hours. The period is 12. Thus,

$$12 = \frac{2\pi}{B} \qquad \text{Our goal is to solve this equation for } B.$$

$$12B = 2\pi \qquad \text{Multiply both sides by } B.$$

$$B = \frac{2\pi}{12} = \frac{\pi}{6}. \qquad \text{Divide both sides by 12.}$$

To find C, we use the phase shift. The blue portion of the graph shows that the starting point of the cycle is shifted from 0 to 7. The phase shift, $\frac{C}{B}$, is 7.

$$7 = \frac{C}{B} \qquad \text{The phase shift of } y = A\sin(Bx - C) \text{ is } \frac{C}{B}.$$

$$7 = \frac{C}{\frac{\pi}{6}} \qquad \text{From above, we have } B = \frac{\pi}{6}.$$

$$\frac{7\pi}{6} = C \qquad \text{Multiply both sides of the equation by } \frac{\pi}{6}.$$

We see that $A = 4$, $B = \frac{\pi}{6}$, $C = \frac{7\pi}{6}$, and $D = 9$. Substitute these values into $y = A\sin(Bx - C) + D$. The water's depth, in feet, x hours after midnight is modeled by

$$y = 4\sin\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 9.$$

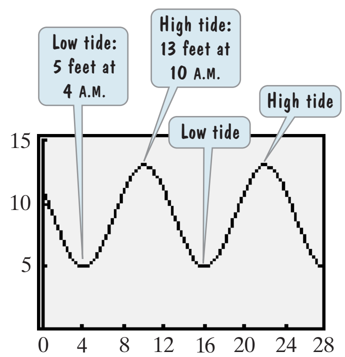
Technology

Graphic Connections

We can use a graphing utility to verify that the model in Example 9,

$$y = 4\sin\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 9,$$

is correct. The graph of the function is shown in a [0, 28, 4] by [0, 15, 5] viewing rectangle.



Check Point 9 A region that is 30° north of the Equator averages a minimum of 10 hours of daylight in December. Hours of daylight are at a maximum of 14 hours in June. Let x represent the month of the year, with 1 for January, 2 for February, 3 for March, and 12 for December. If y represents the number of hours of daylight in month x, use a sine function of the form $y = A\sin(Bx - C) + D$ to model the hours of daylight.

Exercise Set 4.5

Practice Exercises

In Exercises 1–6, determine the amplitude of each function. Then graph the function and $y = \sin x$ in the same rectangular coordinate system for $0 \le x \le 2\pi$.

1. $y = 4\sin x$
2. $y = 5\sin x$
3. $y = \frac{1}{3}\sin x$
4. $y = \frac{1}{4}\sin x$
5. $y = -3\sin x$
6. $y = -4\sin x$

In Exercises 7–16, determine the amplitude and period of each function. Then graph one period of the function.

7. $y = \sin 2x$
8. $y = \sin 4x$
9. $y = 3\sin\frac{1}{2}x$
10. $y = 2\sin\frac{1}{4}x$
11. $y = 4\sin \pi x$
12. $y = 3\sin 2\pi x$
13. $y = -3\sin 2\pi x$
14. $y = -2\sin \pi x$
15. $y = -\sin\frac{2}{3}x$
16. $y = -\sin\frac{4}{3}x$

In Exercises 17–30, determine the amplitude, period, and phase shift of each function. Then graph one period of the function.

17. $y = \sin(x - \pi)$
18. $y = \sin\left(x - \frac{\pi}{2}\right)$
19. $y = \sin(2x - \pi)$
20. $y = \sin\left(2x - \frac{\pi}{2}\right)$
21. $y = 3\sin(2x - \pi)$
22. $y = 3\sin\left(2x - \frac{\pi}{2}\right)$
23. $y = \frac{1}{2}\sin\left(x + \frac{\pi}{2}\right)$
24. $y = \frac{1}{2}\sin(x + \pi)$
25. $y = -2\sin\left(2x + \frac{\pi}{2}\right)$
26. $y = -3\sin\left(2x + \frac{\pi}{2}\right)$
27. $y = 3\sin(\pi x + 2)$
28. $y = 3\sin(2\pi x + 4)$
29. $y = -2\sin(2\pi x + 4\pi)$
30. $y = -3\sin(2\pi x + 4\pi)$

In Exercises 31–34, determine the amplitude of each function. Then graph the function and $y = \cos x$ in the same rectangular coordinate system for $0 \le x \le 2\pi$.

31. $y = 2\cos x$
32. $y = 3\cos x$
33. $y = -2\cos x$
34. $y = -3\cos x$

In Exercises 35–42, determine the amplitude and period of each function. Then graph one period of the function.

35. $y = \cos 2x$
36. $y = \cos 4x$
37. $y = 4\cos 2\pi x$
38. $y = 5\cos 2\pi x$
39. $y = -4\cos\frac{1}{2}x$
40. $y = -3\cos\frac{1}{3}x$
41. $y = -\frac{1}{2}\cos\frac{\pi}{3}x$
42. $y = -\frac{1}{2}\cos\frac{\pi}{4}x$

In Exercises 43–52, determine the amplitude, period, and phase shift of each function. Then graph one period of the function.

43. $y = \cos\left(x - \frac{\pi}{2}\right)$
44. $y = \cos\left(x + \frac{\pi}{2}\right)$
45. $y = 3\cos(2x - \pi)$
46. $y = 4\cos(2x - \pi)$
47. $y = \frac{1}{2}\cos\left(3x + \frac{\pi}{2}\right)$
48. $y = \frac{1}{2}\cos(2x + \pi)$
49. $y = -3\cos\left(2x - \frac{\pi}{2}\right)$
50. $y = -4\cos\left(2x - \frac{\pi}{2}\right)$
51. $y = 2\cos(2\pi x + 8\pi)$
52. $y = 3\cos(2\pi x + 4\pi)$

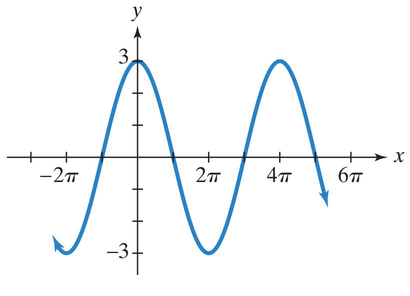
In Exercises 53–60, use a vertical shift to graph one period of the function.

53. $y = \sin x + 2$
54. $y = \sin x - 2$
55. $y = \cos x - 3$
56. $y = \cos x + 3$
57. $y = 2\sin\frac{1}{2}x + 1$
58. $y = 2\cos\frac{1}{2}x + 1$
59. $y = -3\cos 2\pi x + 2$
60. $y = -3\sin 2\pi x + 2$

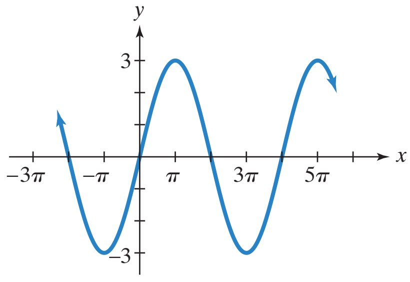
Practice Plus

In Exercises 61–66, find an equation for each graph.

61.



62.



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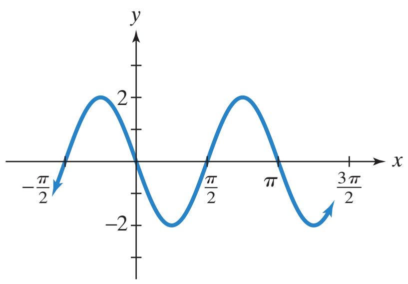

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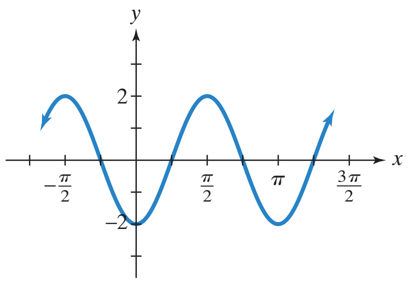

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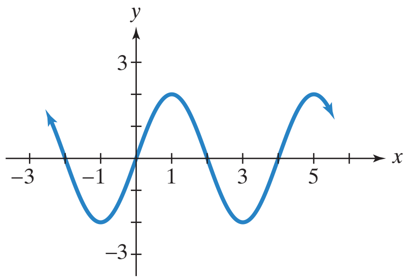
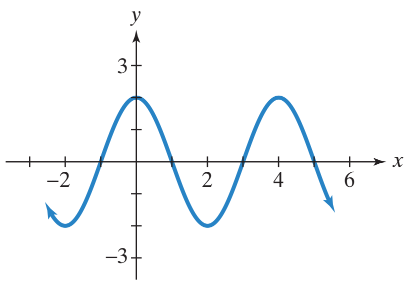


66.



In Exercises 67–70, graph one period of each function.

67. $y = \left|2 \cos \frac{x}{2}\right|$

68. $y = \left|3 \cos \frac{2x}{3}\right|$

69. $y = -|3 \sin \pi x|$

70. $y = -\left|2 \sin \frac{\pi x}{2}\right|$

In Exercises 71–74, graph f, g, and h in the same rectangular coordinate system for $0 \le x \le 2\pi$. Obtain the graph of h by adding or subtracting the corresponding y-coordinates on the graphs of f and g.

71. $f(x) = -2 \sin x,\ g(x) = \sin 2x,\ h(x) = (f + g)(x)$

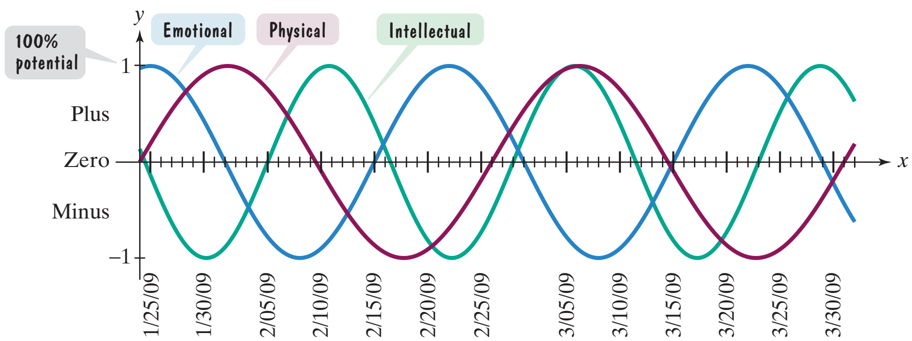
72. $f(x) = 2 \cos x,\ g(x) = \cos 2x,\ h(x) = (f + g)(x)$

73. $f(x) = \sin x,\ g(x) = \cos 2x,\ h(x) = (f - g)(x)$

74. $f(x) = \cos x,\ g(x) = \sin 2x,\ h(x) = (f - g)(x)$

Application Exercises

In the theory of biorhythms, sine functions are used to measure a person's potential. You can obtain your biorhythm chart online by simply entering your date of birth, the date you want your biorhythm chart to begin, and the number of months you wish to be included in the plot. Shown below is your author's chart, beginning January 25, 2009, when he was 23,283 days old. We all have cycles with the same amplitudes and periods as those shown here. Each of our three basic cycles begins at birth. Use the biorhythm chart shown to solve Exercises 75–82. The longer tick marks correspond to the dates shown.



75. What is the period of the physical cycle?

76. What is the period of the emotional cycle?

77. What is the period of the intellectual cycle?

78. For the period shown, what is the worst day in February for your author to run in a marathon?

79. For the period shown, what is the best day in March for your author to meet an online friend for the first time?

80. For the period shown, what is the best day in February for your author to begin writing this trigonometry chapter?

81. If you extend these sinusoidal graphs to the end of the year, is there a day when your author should not even bother getting out of bed?

82. If you extend these sinusoidal graphs to the end of the year, are there any days where your author is at near-peak physical, emotional, and intellectual potential?

83. Rounded to the nearest hour, Los Angeles averages 14 hours of daylight in June, 10 hours in December, and 12 hours in March and September. Let x represent the number of months after June and let y represent the number of hours of daylight in month x. Make a graph that displays the information from June of one year to June of the following year.

84. A clock with an hour hand that is 15 inches long is hanging on a wall. At noon, the distance between the tip of the hour hand and the ceiling is 23 inches. At 3 P.M., the distance is 38 inches; at 6 P.M., 53 inches; at 9 P.M., 38 inches; and at midnight the distance is again 23 inches. If y represents the distance between the tip of the hour hand and the ceiling x hours after noon, make a graph that displays the information for $0 \le x \le 24$.

85. The number of hours of daylight in Boston is given by

$$y = 3\sin\frac{2\pi}{365}(x - 79) + 12,$$

where x is the number of days after January 1.

a. What is the amplitude of this function?

b. What is the period of this function?

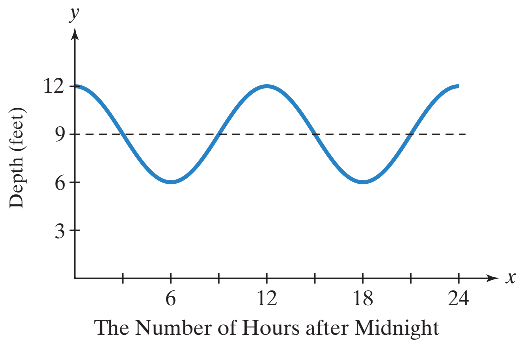
c. How many hours of daylight are there on the longest day of the year?

d. How many hours of daylight are there on the shortest day of the year?

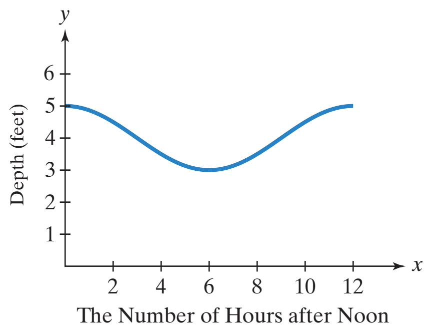
e. Graph the function for one period, starting on January 1.

86. The average monthly temperature, y, in degrees Fahrenheit, for Juneau, Alaska, can be modeled by $y = 16\sin\left(\frac{\pi}{6}x - \frac{2\pi}{3}\right) + 40$, where x is the month of the year (January = 1, February = 2, ... December = 12). Graph the function for $1 \le x \le 12$. What is the highest average monthly temperature? In which month does this occur?

87. The figure shows the depth of water at the end of a boat dock. The depth is 6 feet at low tide and 12 feet at high tide. On a certain day, low tide occurs at 6 A.M. and high tide at noon. If y represents the depth of the water x hours after midnight, use a cosine function of the form $y = A\cos Bx + D$ to model the water's depth.



88. The figure in the next column shows the depth of water at the end of a boat dock. The depth is 5 feet at high tide and 3 feet at low tide. On a certain day, high tide occurs at noon and low tide at 6 P.M. If y represents the depth of the water x hours after noon, use a cosine function of the form $y = A\cos Bx + D$ to model the water's depth.



Writing in Mathematics

89. Without drawing a graph, describe the behavior of the basic sine curve.

90. What is the amplitude of the sine function? What does this tell you about the graph?

91. If you are given the equation of a sine function, how do you determine the period?

92. What does a phase shift indicate about the graph of a sine function? How do you determine the phase shift from the function's equation?

93. Describe a general procedure for obtaining the graph of $y = A\sin(Bx - C)$.

94. Without drawing a graph, describe the behavior of the basic cosine curve.

95. Describe a relationship between the graphs of $y = \sin x$ and $y = \cos x$.

96. Describe the relationship between the graphs of $y = A\cos(Bx - C)$ and $y = A\cos(Bx - C) + D$.

97. Biorhythm cycles provide interesting applications of sinusoidal graphs. But do you believe in the validity of biorhythms? Write a few sentences explaining why or why not.

Technology Exercises

98. Use a graphing utility to verify any five of the sine curves that you drew by hand in Exercises 7–30. The amplitude, period, and phase shift should help you to determine appropriate viewing rectangle settings.

99. Use a graphing utility to verify any five of the cosine curves that you drew by hand in Exercises 35–52.

100. Use a graphing utility to verify any two of the sinusoidal curves with vertical shifts that you drew in Exercises 53–60.

In Exercises 101–104, use a graphing utility to graph two periods of the function.

101. $y = 3\sin(2x + \pi)$

102. $y = -2\cos\left(2\pi x - \frac{\pi}{2}\right)$

103. $y = 0.2\sin\left(\frac{\pi}{10}x + \pi\right)$

104. $y = 3\sin(2x - \pi) + 5$

105. Use a graphing utility to graph $y = \sin x$ and $y = x - \frac{x^3}{6} + \frac{x^5}{120}$ in a $\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$ viewing rectangle. How do the graphs compare?

106. Use a graphing utility to graph $y = \cos x$ and $y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ in a $\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$ viewing rectangle. How do the graphs compare?

107. Use a graphing utility to graph

$$y = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4}$$

in a $\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$ viewing rectangle. How do these waves compare to the smooth rolling waves of the basic sine curve?

108. Use a graphing utility to graph

$$y = \sin x - \frac{\sin 3x}{9} + \frac{\sin 5x}{25}$$

in a $\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$ viewing rectangle. How do these waves compare to the smooth rolling waves of the basic sine curve?

109. The data show the average monthly temperatures for Washington, D.C.

x (Month)		**Average Monthly Temperature, °F**
1	(January)	**34.6**
2	(February)	**37.5**
3	(March)	**47.2**
4	(April)	**56.5**
5	(May)	**66.4**
6	(June)	**75.6**
7	(July)	**80.0**
8	(August)	**78.5**
9	(September)	**71.3**
10	(October)	**59.7**
11	(November)	**49.8**
12	(December)	**39.4**

Source: U.S. National Oceanic and Atmospheric Administration

a. Use your graphing utility to draw a scatter plot of the data from $x = 1$ through $x = 12$.

b. Use the SINe REGression feature to find the sinusoidal function of the form $y = A\sin(Bx + C) + D$ that best fits the data.

c. Use your graphing utility to draw the sinusoidal function of best fit on the scatter plot.

110. Repeat Exercise 109 for data of your choice. The data can involve the average monthly temperatures for the region where you live or any data whose scatter plot takes the form of a sinusoidal function.

Critical Thinking Exercises

Make Sense? *In Exercises 111–114, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

111. When graphing one complete cycle of $y = A\sin(Bx - C)$, I find it easiest to begin my graph on the x-axis.

112. When graphing one complete cycle of $y = A\cos(Bx - C)$, I find it easiest to begin my graph on the x-axis.

113. Using the equation $y = A\sin Bx$, if I replace either A or B with its opposite, the graph of the resulting equation is a reflection of the graph of the original equation about the x-axis.

114. A ride on a circular Ferris wheel is like riding sinusoidal graphs.

115. Determine the range of each of the following functions. Then give a viewing rectangle, or window, that shows two periods of the function's graph.

a. $f(x) = 3\sin\left(x + \frac{\pi}{6}\right) - 2$

b. $g(x) = \sin 3\left(x + \frac{\pi}{6}\right) - 2$

116. Write the equation for a cosine function with amplitude π, period 1, and phase shift -2.

In Chapter 5, we will prove the following identities:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x.$$

Use these identities to solve Exercises 117–118.

117. Use the identity for $\sin^2 x$ to graph one period of $y = \sin^2 x$.

118. Use the identity for $\cos^2 x$ to graph one period of $y = \cos^2 x$.

Group Exercise

119. This exercise is intended to provide some fun with biorhythms, regardless of whether you believe they have any validity. We will use each member's chart to determine biorhythmic compatibility. Before meeting, each group member should go online and obtain his or her biorhythm chart. The date of the group meeting is the date on which your chart should begin. Include 12 months in the plot. At the meeting, compare differences and similarities among the intellectual sinusoidal curves. Using these comparisons, each person should find the one other person with whom he or she would be most intellectually compatible.

Preview Exercises

Exercises 120–122 will help you prepare for the material covered in the next section.

120. Solve: $-\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$.

121. Simplify: $\dfrac{-\frac{3\pi}{4} + \frac{\pi}{4}}{2}$.

122. a. Graph $y = -3\cos\frac{x}{2}$ for $-\pi \le x \le 5\pi$.

b. Consider the reciprocal function of $y = -3\cos\frac{x}{2}$, namely, $y = -3\sec\frac{x}{2}$. What does your graph from part (a) indicate about this reciprocal function for $x = -\pi, \pi, 3\pi$, and 5π?