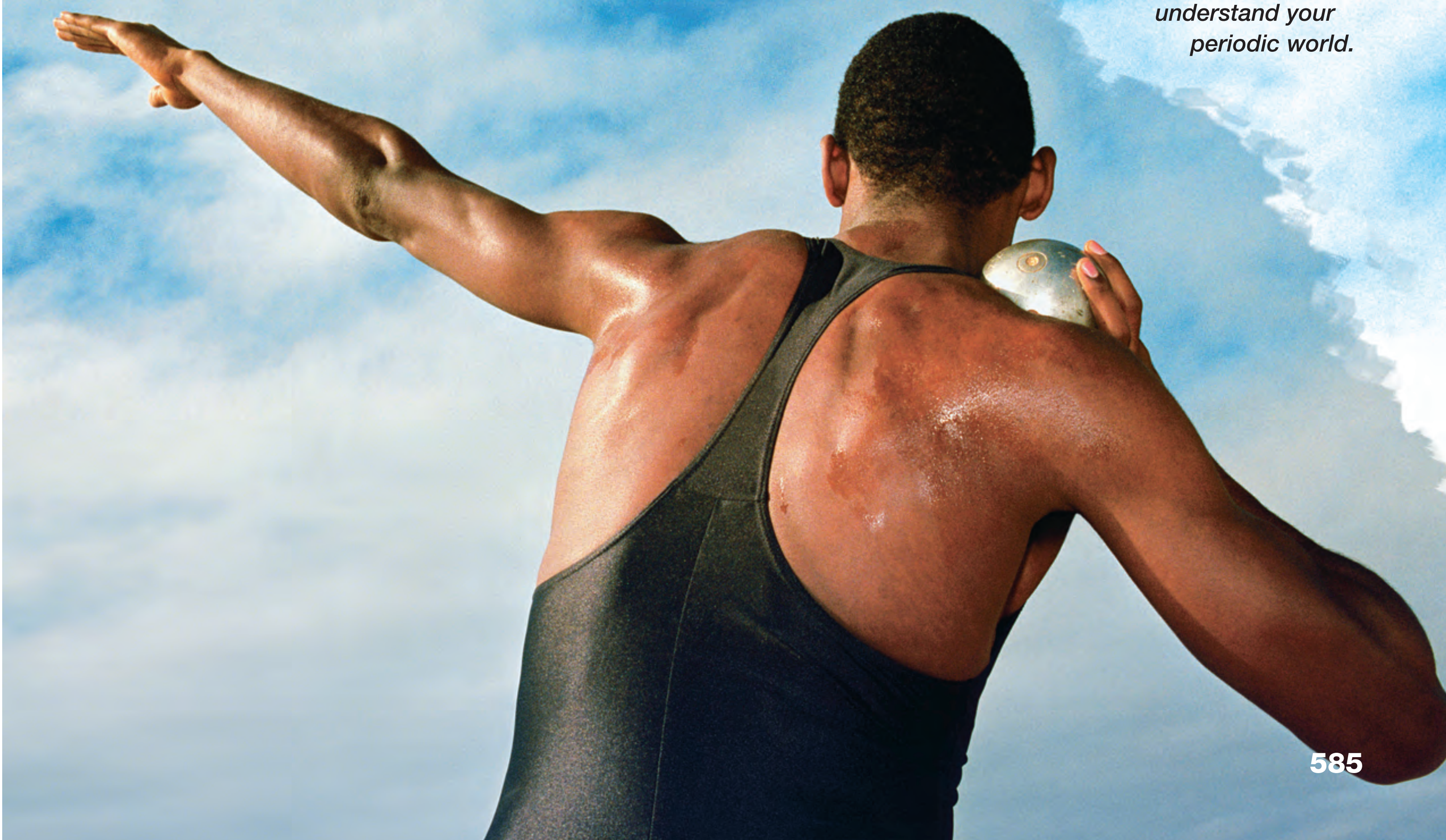
Analytic Trigonometry

5

You enjoy watching your friend participate in the shot put at college track and field events. After a few full turns in a circle, he throws ("puts") an 8-pound, 13-ounce shot from the shoulder. The range of his throwing distance continues to improve. Knowing that you are studying trigonometry, he asks if there is some way that a trigonometric expression might help achieve his best possible distance in the event.

This problem appears as Exercise 79 in Exercise Set 5.3. In the solution, you will obtain critical information about athletic performance using a trigonometric identity. In this chapter, we derive important categories of identities involving trigonometric functions. You will learn how to use these identities to better understand your periodic world.



Section 5.1

Verifying Trigonometric Identities

Objective

1 Use the fundamental trigonometric identities to verify identities.

Do you enjoy solving puzzles? The process is a natural way to develop problem-solving skills that are important in every area of our lives. Engaging in problem solving for sheer pleasure releases chemicals in the brain that enhance our feeling of well-being. Perhaps this is why puzzles have fascinated people for over 12,000 years.

Thousands of relationships exist among the six trigonometric functions. Verifying these relationships is like solving a puzzle. Why? There are no rigid rules for the process. Thus, proving a trigonometric relationship requires you to be creative in your approach to problem solving. By learning to establish these relationships, you will become a better, more confident problem solver. Furthermore, you may enjoy the feeling of satisfaction that accompanies solving each "puzzle."

The Fundamental Identities

In Chapter 4, we used right triangles to establish relationships among the trigonometric functions. Although we limited domains to acute angles, the fundamental identities listed in the following box are true for all values of x for which the expressions are defined.

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin x = \frac{1}{\csc x} \quad \cos x = \frac{1}{\sec x} \quad \tan x = \frac{1}{\cot x}$$

$$\csc x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x} \quad \cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \quad 1 + \tan^2 x = \sec^2 x \quad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \quad \cos(-x) = \cos x \quad \tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x \quad \sec(-x) = \sec x \quad \cot(-x) = -\cot x$$

Study Tip

Memorize the identities in the box. You may need to use variations of these fundamental identities. For example, instead of

$$\sin^2 x + \cos^2 x = 1$$

you might want to use

$$\sin^2 x = 1 - \cos^2 x$$

or

$$\cos^2 x = 1 - \sin^2 x.$$

Therefore, it is important to know each relationship well so that mental algebraic manipulation is possible.

Using Fundamental Identities to Verify Other Identities

1 Use the fundamental trigonometric identities to verify identities.

The fundamental trigonometric identities are used to establish other relationships among trigonometric functions. To **verify an identity**, we show that one side of the identity can be simplified so that it is identical to the other side. Each side of the equation is manipulated independently of the other side of the equation. Start with the side containing the more complicated expression. If you substitute one or more fundamental identities on the more complicated side, you will often be able to rewrite it in a form identical to that of the other side.

No one method or technique can be used to verify every identity. Some identities can be verified by rewriting the more complicated side so that it contains only sines and cosines.

EXAMPLE 1 Changing to Sines and Cosines to Verify an Identity

Verify the identity: $\sec x \cot x = \csc x$.

Solution The left side of the equation contains the more complicated expression. Thus, we work with the left side. Let us express this side of the identity in terms of sines and cosines. Perhaps this strategy will enable us to transform the left side into $\csc x$, the expression on the right.

$$\sec x \cot x = \frac{1}{\cos x} \cdot \frac{\cos x}{\sin x}$$

Apply a reciprocal identity: $\sec x = \frac{1}{\cos x}$ and a quotient identity: $\cot x = \frac{\cos x}{\sin x}$.

$$= \frac{1}{\cancel{\cos x}_1} \cdot \frac{\cancel{\cos x}^1}{\sin x}$$

Divide both the numerator and the denominator by $\cos x$, the common factor.

$$= \frac{1}{\sin x}$$

Multiply the remaining factors in the numerator and denominator.

$$= \csc x$$

Apply a reciprocal identity: $\csc x = \frac{1}{\sin x}$.

By working with the left side and simplifying it so that it is identical to the right side, we have verified the given identity.

Technology

Numeric and Graphic Connections

You can use a graphing utility to provide evidence of an identity. Enter each side of the identity separately under y_1 and y_2. Then use the TABLE feature or the graphs. The table should show that the function values are the same except for those values of x for which y_1, y_2, or both, are undefined. The graphs should appear to be identical.

Let's check the identity in Example 1:

$$\sec x \cot x = \csc x.$$

$y_1 = \sec x \cot x$. Enter $\sec x$ as $\frac{1}{\cos x}$ and $\cot x$ as $\frac{1}{\tan x}$.

$y_2 = \csc x$. Enter $\csc x$ as $\frac{1}{\sin x}$.

Numeric Check

Display a table for y_1 and y_2. We started our table at $-\pi$ and used $\Delta Tbl = \frac{\pi}{8}$.

$y_1 = \sec x \cot x$ $y_2 = \csc x$

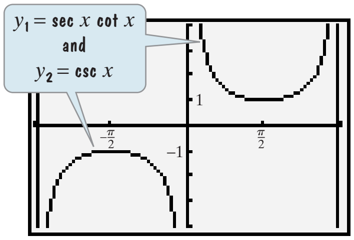
X	Y1	Y2
-3.142	ERROR	ERROR
-2.749	-2.613	-2.613
-2.356	-1.414	-1.414
-1.963	-1.082	-1.082
-1.571	ERROR	-1
-1.178	-1.082	-1.082
-.7854	-1.414	-1.414

X=-3.14159265359

Function values are the same except for values of x for which y_1, y_2, or both, are undefined.

Graphic Check

Display graphs for y_1 and y_2.



$[-\pi, \pi, \frac{\pi}{2}]$ by $[-4, 4, 1]$

The graphs appear to be identical.

Check Point 1 Verify the identity: $\csc x \tan x = \sec x$.

In verifying an identity, stay focused on your goal. When manipulating one side of the equation, continue to look at the other side to keep the desired form of the result in mind.

Study Tip

Verifying that an equation is an identity is different from solving an equation. You do not verify an identity by adding, subtracting, multiplying, or dividing each side by the same expression. If you do this, you have already assumed that the given statement is true. You do not know that it is true until after you have verified it.

EXAMPLE 2 Changing to Sines and Cosines to Verify an Identity

Verify the identity: $\sin x \tan x + \cos x = \sec x$.

Solution The left side is more complicated, so we start with it. Notice that the left side contains the sum of two terms, but the right side contains only one term. This means that somewhere during the verification process, the two terms on the left side must be added to form one term.

Let's begin by expressing the left side of the identity so that it contains only sines and cosines. Thus, we apply a quotient identity and replace $\tan x$ by $\frac{\sin x}{\cos x}$. Perhaps this strategy will enable us to transform the left side into $\sec x$, the expression on the right.

Study Tip

When proving identities, be sure to write the variable associated with each trigonometric function. Do not get lazy and write

$$\sin \tan + \cos$$

for

$$\sin x \tan x + \cos x$$

because sin, tan, and cos are meaningless without specified variables.

$$\sin x \tan x + \cos x = \sin x\left(\frac{\sin x}{\cos x}\right) + \cos x$$
Apply a quotient identity: $\tan x = \frac{\sin x}{\cos x}$.

$$= \frac{\sin^2 x}{\cos x} + \cos x$$
Multiply.

$$= \frac{\sin^2 x}{\cos x} + \cos x \cdot \frac{\cos x}{\cos x}$$
The least common denominator is $\cos x$. Write the second expression with a denominator of $\cos x$.

$$= \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x}$$
Multiply.

$$= \frac{\sin^2 x + \cos^2 x}{\cos x}$$
Add numerators and place this sum over the least common denominator.

$$= \frac{1}{\cos x}$$
Apply a Pythagorean identity: $\sin^2 x + \cos^2 x = 1$.

$$= \sec x$$
Apply a reciprocal identity: $\sec x = \frac{1}{\cos x}$.

By working with the left side and arriving at the right side, the identity is verified.

Check Point 2 Verify the identity: $\cos x \cot x + \sin x = \csc x$.

Some identities are verified using factoring to simplify a trigonometric expression.

EXAMPLE 3 Using Factoring to Verify an Identity

Verify the identity: $\cos x - \cos x \sin^2 x = \cos^3 x$.

Solution We start with the more complicated side, the left side. Factor out the greatest common factor, $\cos x$, from each of the two terms.

$$\cos x - \cos x \sin^2 x = \cos x(1 - \sin^2 x)$$ Factor $\cos x$ from the two terms.

$$= \cos x \cdot \cos^2 x$$ Use a variation of $\sin^2 x + \cos^2 x = 1$. Solving for $\cos^2 x$, we obtain $\cos^2 x = 1 - \sin^2 x$.

$$= \cos^3 x$$ Multiply.

We worked with the left side and arrived at the right side. Thus, the identity is verified.

Check Point 3 Verify the identity: $\sin x - \sin x \cos^2 x = \sin^3 x$.

There is often more than one technique that can be used to verify an identity.

EXAMPLE 4 Using Two Techniques to Verify an Identity

Verify the identity: $\dfrac{1 + \sin\theta}{\cos\theta} = \sec\theta + \tan\theta$.

Solution

Method 1. Separating a Single-Term Quotient into Two Terms

Let's separate the quotient on the left side into two terms using

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}.$$

Perhaps this strategy will enable us to transform the left side into $\sec\theta + \tan\theta$, the sum on the right.

$$\frac{1 + \sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$ Divide each term in the numerator by $\cos\theta$.

$$= \sec\theta + \tan\theta$$ Apply a reciprocal identity and a quotient identity: $\sec\theta = \dfrac{1}{\cos\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

We worked with the left side and arrived at the right side. Thus, the identity is verified.

Method 2. Changing to Sines and Cosines

Let's work with the right side of the identity and express it so that it contains only sines and cosines.

$$\sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$ Apply a reciprocal identity and a quotient identity: $\sec\theta = \dfrac{1}{\cos\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$= \frac{1 + \sin\theta}{\cos\theta}$$ Add numerators. Put this sum over the common denominator.

We worked with the right side and arrived at the left side. Thus, the identity is verified.

Check Point 4 Verify the identity: $\dfrac{1 + \cos\theta}{\sin\theta} = \csc\theta + \cot\theta$.

How do we verify identities in which sums or differences of fractions with trigonometric functions appear on one side? Use the least common denominator and combine the fractions. This technique is especially useful when the other side of the identity contains only one term.

EXAMPLE 5 Combining Fractional Expressions to Verify an Identity

Verify the identity: $\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} = 2 \sec x.$

Solution We start with the more complicated side, the left side. The least common denominator of the fractions is $(1 + \sin x)(\cos x)$. We express each fraction in terms of this least common denominator by multiplying the numerator and denominator by the extra factor needed to form $(1 + \sin x)(\cos x)$.

$$\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x}$$

The least common denominator is $(1 + \sin x)(\cos x)$.

$$= \frac{\cos x(\cos x)}{(1 + \sin x)(\cos x)} + \frac{(1 + \sin x)(1 + \sin x)}{(1 + \sin x)(\cos x)}$$

Rewrite each fraction with the least common denominator.

$$= \frac{\cos^2 x}{(1 + \sin x)(\cos x)} + \frac{1 + 2 \sin x + \sin^2 x}{(1 + \sin x)(\cos x)}$$

Use the FOIL method to multiply $(1 + \sin x)(1 + \sin x)$.

$$= \frac{\cos^2 x + 1 + 2 \sin x + \sin^2 x}{(1 + \sin x)(\cos x)}$$

Add numerators. Put this sum over the least common denominator.

$$= \frac{(\sin^2 x + \cos^2 x) + 1 + 2 \sin x}{(1 + \sin x)(\cos x)}$$

Regroup terms to apply a Pythagorean identity.

$$= \frac{1 + 1 + 2 \sin x}{(1 + \sin x)(\cos x)}$$

Apply a Pythagorean identity: $\sin^2 x + \cos^2 x = 1$.

$$= \frac{2 + 2 \sin x}{(1 + \sin x)(\cos x)}$$

Add constant terms in the numerator: $1 + 1 = 2$.

$$= \frac{2\cancel{(1 + \sin x)}}{\cancel{(1 + \sin x)}(\cos x)}$$

Factor and simplify.

$$= \frac{2}{\cos x}$$

$$= 2 \sec x$$

Apply a reciprocal identity: $\sec x = \frac{1}{\cos x}$.

We worked with the left side and arrived at the right side. Thus, the identity is verified.

Study Tip

Some students have difficulty verifying identities due to problems working with fractions. If this applies to you, review the section on rational expressions in Chapter P.

Check Point 5 Verify the identity: $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = 2 \csc x.$

Some identities are verified using a technique that may remind you of rationalizing a denominator.

EXAMPLE 6 Multiplying the Numerator and Denominator by the Same Factor to Verify an Identity

Verify the identity: $\frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}.$

Solution The suggestions given in the previous examples do not apply here. Everything is already expressed in terms of sines and cosines. Furthermore, there are no fractions to combine and neither side looks more complicated than the other. Let's solve the puzzle by working with the left side and making it look like the expression on the right. The expression on the right contains $1 - \cos x$ in the numerator. This suggests multiplying the numerator and denominator of the left side by $1 - \cos x$. By doing this, we obtain a factor of $1 - \cos x$ in the numerator, as in the numerator on the right.

$$\frac{\sin x}{1+\cos x} = \frac{\sin x}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x}$$ Multiply numerator and denominator by $1 - \cos x$.

$$= \frac{\sin x(1-\cos x)}{1-\cos^2 x}$$ Multiply. Use $(A+B)(A-B) = A^2 - B^2$, with $A = 1$ and $B = \cos x$, to multiply denominators.

$$= \frac{\sin x(1-\cos x)}{\sin^2 x}$$ Use a variation of $\sin^2 x + \cos^2 x = 1$. Solving for $\sin^2 x$, we obtain $\sin^2 x = 1 - \cos^2 x$.

$$= \frac{1-\cos x}{\sin x}$$ Simplify: $\frac{\sin x}{\sin^2 x} = \frac{\sin x}{\sin x \cdot \sin x} = \frac{1}{\sin x}$.

We worked with the left side and arrived at the right side. Thus, the identity is verified.

Discovery

Verify the identity in Example 6 by making the right side look like the left side. Start with the expression on the right. Multiply the numerator and denominator by $1 + \cos x$.

Check Point 6 Verify the identity: $\frac{\cos x}{1+\sin x} = \frac{1-\sin x}{\cos x}$.

EXAMPLE 7 Changing to Sines and Cosines to Verify an Identity

Verify the identity: $\frac{\tan x - \sin(-x)}{1+\cos x} = \tan x$.

Solution We begin with the left side. Our goal is to obtain $\tan x$, the expression on the right.

$$\frac{\tan x - \sin(-x)}{1+\cos x} = \frac{\tan x - (-\sin x)}{1+\cos x}$$ The sine function is odd: $\sin(-x) = -\sin x$.

$$= \frac{\tan x + \sin x}{1+\cos x}$$ Simplify.

$$= \frac{\frac{\sin x}{\cos x} + \sin x}{1+\cos x}$$ Apply a quotient identity: $\tan x = \frac{\sin x}{\cos x}$.

$$= \frac{\frac{\sin x}{\cos x} + \frac{\sin x \cos x}{\cos x}}{1+\cos x}$$ Express the terms in the numerator with the least common denominator, $\cos x$.

$$= \frac{\frac{\sin x + \sin x \cos x}{\cos x}}{1+\cos x}$$ Add in the numerator.

$$= \frac{\sin x + \sin x \cos x}{\cos x} \div \frac{1+\cos x}{1}$$ Rewrite the main fraction bar as $\div$.

$$= \frac{\sin x + \sin x \cos x}{\cos x} \cdot \frac{1}{1+\cos x}$$ Invert the divisor and multiply.

$$= \frac{\sin x(1+\cos x)}{\cos x} \cdot \frac{1}{1+\cos x}$$ Factor and simplify.

$$= \frac{\sin x}{\cos x}$$ Multiply the remaining factors in the numerator and in the denominator.

$$= \tan x$$ Apply a quotient identity.

The left side simplifies to $\tan x$, the right side. Thus, the identity is verified.

Discovery

Try simplifying

$$\frac{\frac{\sin x}{\cos x} + \sin x}{1+\cos x}$$

by multiplying the two terms in the numerator and the two terms in the denominator by $\cos x$. This method for simplifying the complex fraction involves multiplying the numerator and the denominator by the least common denominator of all fractions in the expression. Do you prefer this simplification procedure over the method used on the right?

Check Point 7 Verify the identity: $\dfrac{\sec x + \csc(-x)}{\sec x \csc x} = \sin x - \cos x$.

Is every identity verified by working with only one side? No. You can sometimes work with each side separately and show that both sides are equal to the same trigonometric expression. This is illustrated in Example 8.

EXAMPLE 8 Working with Both Sides Separately to Verify an Identity

Verify the identity: $\dfrac{1}{1 + \cos\theta} + \dfrac{1}{1 - \cos\theta} = 2 + 2\cot^2\theta$.

Solution We begin by working with the left side.

$$\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta}$$
The least common denominator is $(1 + \cos\theta)(1 - \cos\theta)$.

$$= \frac{1(1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)} + \frac{1(1 + \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)}$$
Rewrite each fraction with the least common denominator.

$$= \frac{1 - \cos\theta + 1 + \cos\theta}{(1 + \cos\theta)(1 - \cos\theta)}$$
Add numerators. Put this sum over the least common denominator.

$$= \frac{2}{(1 + \cos\theta)(1 - \cos\theta)}$$
Simplify the numerator: $-\cos\theta + \cos\theta = 0$ and $1 + 1 = 2$.

$$= \frac{2}{1 - \cos^2\theta}$$
Multiply the factors in the denominator.

Now we work with the right side. Our goal is to transform this side into the simplified form attained for the left side, $\dfrac{2}{1 - \cos^2\theta}$.

$$2 + 2\cot^2\theta = 2 + 2\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$
Use a quotient identity: $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

$$= \frac{2\sin^2\theta}{\sin^2\theta} + \frac{2\cos^2\theta}{\sin^2\theta}$$
Rewrite each term with the least common denominator, $\sin^2\theta$.

$$= \frac{2\sin^2\theta + 2\cos^2\theta}{\sin^2\theta}$$
Add numerators. Put this sum over the least common denominator.

$$= \frac{2(\sin^2\theta + \cos^2\theta)}{\sin^2\theta}$$
Factor out the greatest common factor, 2.

$$= \frac{2}{\sin^2\theta}$$
Apply a Pythagorean identity: $\sin^2\theta + \cos^2\theta = 1$.

$$= \frac{2}{1 - \cos^2\theta}$$
Use a variation of $\sin^2\theta + \cos^2\theta = 1$ and solve for $\sin^2\theta$: $\sin^2\theta = 1 - \cos^2\theta$.

The identity is verified because both sides are equal to $\dfrac{2}{1 - \cos^2\theta}$.

Check Point 8 Verify the identity: $\dfrac{1}{1 + \sin\theta} + \dfrac{1}{1 - \sin\theta} = 2 + 2\tan^2\theta$.

Guidelines for Verifying Trigonometric Identities

There is often more than one correct way to solve a puzzle, although one method may be shorter and more efficient than another. The same is true for verifying an identity. For example, how would you verify

$$\frac{\csc^2 x - 1}{\csc^2 x} = \cos^2 x?$$

One approach is to use a Pythagorean identity, $1 + \cot^2 x = \csc^2 x$, on the left side. Then change the resulting expression to sines and cosines.

$$\frac{\csc^2 x - 1}{\csc^2 x} = \frac{(1 + \cot^2 x) - 1}{\csc^2 x} = \frac{\cot^2 x}{\csc^2 x} = \frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{1}{\sin^2 x}} = \frac{\cos^2 x}{\sin^2 x} \cdot \frac{\sin^2 x}{1} = \cos^2 x$$

Apply a Pythagorean identity: $1 + \cot^2 x = \csc^2 x$.

Use $\cot x = \frac{\cos x}{\sin x}$ and $\csc x = \frac{1}{\sin x}$ to change to sines and cosines.

Invert the divisor and multiply.

A more efficient strategy for verifying this identity may not be apparent at first glance. Work with the left side and divide each term in the numerator by the denominator, $\csc^2 x$.

$$\frac{\csc^2 x - 1}{\csc^2 x} = \frac{\csc^2 x}{\csc^2 x} - \frac{1}{\csc^2 x} = 1 - \sin^2 x = \cos^2 x$$

Apply a reciprocal identity: $\sin x = \frac{1}{\csc x}$.

Use $\sin^2 x + \cos^2 x = 1$ and solve for $\cos^2 x$.

With this strategy, we again obtain $\cos^2 x$, the expression on the right side, and it takes fewer steps than the first approach.

An even longer strategy, but one that works, is to replace each of the two occurrences of $\csc^2 x$ on the left side by $\frac{1}{\sin^2 x}$. This may be the approach that you first consider, particularly if you become accustomed to rewriting the more complicated side in terms of sines and cosines. The selection of an appropriate fundamental identity to solve the puzzle most efficiently is learned through lots of practice.

The more identities you prove, the more confident and efficient you will become. Although practice is the only way to learn how to verify identities, there are some guidelines developed throughout the section that should help you get started.

Guidelines for Verifying Trigonometric Identities

- Work with each side of the equation independently of the other side. Start with the more complicated side and transform it in a step-by-step fashion until it looks exactly like the other side.
- Analyze the identity and look for opportunities to apply the fundamental identities.
- Try using one or more of the following techniques:
  1. Rewrite the more complicated side in terms of sines and cosines.
  2. Factor out the greatest common factor.
  3. Separate a single-term quotient into two terms:
  $$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad \text{and} \quad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}.$$
  4. Combine fractional expressions using the least common denominator.
  5. Multiply the numerator and the denominator by a binomial factor that appears on the other side of the identity.
- Don't be afraid to stop and start over again if you are not getting anywhere. Creative puzzle solvers know that strategies leading to dead ends often provide good problem-solving ideas.

Exercise Set 5.1

Practice Exercises

In Exercises 1–60, verify each identity.

1. $\sin x \sec x = \tan x$

2. $\cos x \csc x = \cot x$

3. $\tan(-x)\cos x = -\sin x$

4. $\cot(-x)\sin x = -\cos x$

5. $\tan x \csc x \cos x = 1$

6. $\cot x \sec x \sin x = 1$

7. $\sec x - \sec x \sin^2 x = \cos x$

8. $\csc x - \csc x \cos^2 x = \sin x$

9. $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

10. $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$

11. $\csc\theta - \sin\theta = \cot\theta\cos\theta$

12. $\tan\theta + \cot\theta = \sec\theta\csc\theta$

13. $\dfrac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

14. $\dfrac{\cos\theta\sec\theta}{\cot\theta} = \tan\theta$

15. $\sin^2\theta(1 + \cot^2\theta) = 1$

16. $\cos^2\theta(1 + \tan^2\theta) = 1$

17. $\sin t \tan t = \dfrac{1 - \cos^2 t}{\cos t}$

18. $\cos t \cot t = \dfrac{1 - \sin^2 t}{\sin t}$

19. $\dfrac{\csc^2 t}{\cot t} = \csc t \sec t$

20. $\dfrac{\sec^2 t}{\tan t} = \sec t \csc t$

21. $\dfrac{\tan^2 t}{\sec t} = \sec t - \cos t$

22. $\dfrac{\cot^2 t}{\csc t} = \csc t - \sin t$

23. $\dfrac{1 - \cos\theta}{\sin\theta} = \csc\theta - \cot\theta$

24. $\dfrac{1 - \sin\theta}{\cos\theta} = \sec\theta - \tan\theta$

25. $\dfrac{\sin t}{\csc t} + \dfrac{\cos t}{\sec t} = 1$

26. $\dfrac{\sin t}{\tan t} + \dfrac{\cos t}{\cot t} = \sin t + \cos t$

27. $\tan t + \dfrac{\cos t}{1 + \sin t} = \sec t$

28. $\cot t + \dfrac{\sin t}{1 + \cos t} = \csc t$

29. $1 - \dfrac{\sin^2 x}{1 + \cos x} = \cos x$

30. $1 - \dfrac{\cos^2 x}{1 + \sin x} = \sin x$

31. $\dfrac{\cos x}{1 - \sin x} + \dfrac{1 - \sin x}{\cos x} = 2\sec x$

32. $\dfrac{\sin x}{\cos x + 1} + \dfrac{\cos x - 1}{\sin x} = 0$

33. $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$

34. $\csc^2 x \sec x = \sec x + \csc x \cot x$

35. $\dfrac{\sec x - \csc x}{\sec x + \csc x} = \dfrac{\tan x - 1}{\tan x + 1}$

36. $\dfrac{\csc x - \sec x}{\csc x + \sec x} = \dfrac{\cot x - 1}{\cot x + 1}$

37. $\dfrac{\sin^2 x - \cos^2 x}{\sin x + \cos x} = \sin x - \cos x$

38. $\dfrac{\tan^2 x - \cot^2 x}{\tan x + \cot x} = \tan x - \cot x$

39. $\tan^2 2x + \sin^2 2x + \cos^2 2x = \sec^2 2x$

40. $\cot^2 2x + \cos^2 2x + \sin^2 2x = \csc^2 2x$

41. $\dfrac{\tan 2\theta + \cot 2\theta}{\csc 2\theta} = \sec 2\theta$

42. $\dfrac{\tan 2\theta + \cot 2\theta}{\sec 2\theta} = \csc 2\theta$

43. $\dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$

44. $\dfrac{\cot x + \cot y}{1 - \cot x \cot y} = \dfrac{\cos x \sin y + \sin x \cos y}{\sin x \sin y - \cos x \cos y}$

45. $(\sec x - \tan x)^2 = \dfrac{1 - \sin x}{1 + \sin x}$

46. $(\csc x - \cot x)^2 = \dfrac{1 - \cos x}{1 + \cos x}$

47. $\dfrac{\sec t + 1}{\tan t} = \dfrac{\tan t}{\sec t - 1}$

48. $\dfrac{\csc t - 1}{\cot t} = \dfrac{\cot t}{\csc t + 1}$

49. $\dfrac{1 + \cos t}{1 - \cos t} = (\csc t + \cot t)^2$

50. $\dfrac{\cos^2 t + 4\cos t + 4}{\cos t + 2} = \dfrac{2\sec t + 1}{\sec t}$

51. $\cos^4 t - \sin^4 t = 1 - 2\sin^2 t$

52. $\sin^4 t - \cos^4 t = 1 - 2\cos^2 t$

53. $\dfrac{\sin\theta - \cos\theta}{\sin\theta} + \dfrac{\cos\theta - \sin\theta}{\cos\theta} = 2 - \sec\theta\csc\theta$

54. $\dfrac{\sin\theta}{1 - \cot\theta} - \dfrac{\cos\theta}{\tan\theta - 1} = \sin\theta + \cos\theta$

55. $(\tan^2\theta + 1)(\cos^2\theta + 1) = \tan^2\theta + 2$

56. $(\cot^2\theta + 1)(\sin^2\theta + 1) = \cot^2\theta + 2$

57. $(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2 = 2$

58. $(3\cos\theta - 4\sin\theta)^2 + (4\cos\theta + 3\sin\theta)^2 = 25$

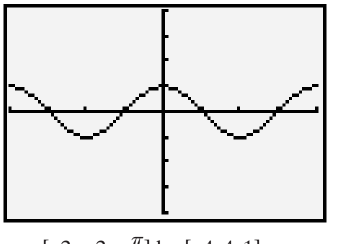
59. $\dfrac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} = \cos^2 x$

60. $\dfrac{\sin x + \cos x}{\sin x} - \dfrac{\cos x - \sin x}{\cos x} = \sec x \csc x$

Practice Plus

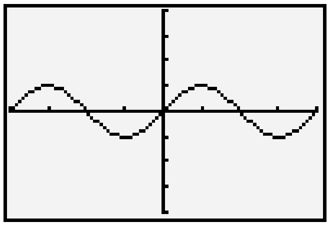
In Exercises 61–66, half of an identity and the graph of this half are given. Use the graph to make a conjecture as to what the right side of the identity should be. Then prove your conjecture.

61. $\dfrac{(\sec x + \tan x)(\sec x - \tan x)}{\sec x} = ?$



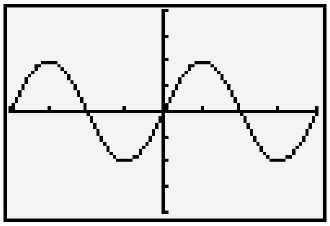
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-4, 4, 1]$

62. $\dfrac{\sec^2 x \csc x}{\sec^2 x + \csc^2 x} = ?$



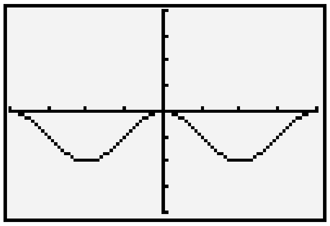
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-4, 4, 1]$

63. $\dfrac{\cos x + \cot x \sin x}{\cot x} = ?$



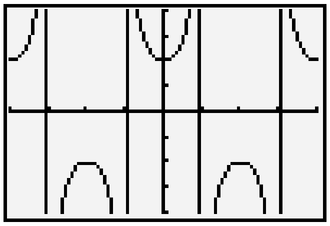
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-4, 4, 1]$

64. $\dfrac{\cos x \tan x - \tan x + 2 \cos x - 2}{\tan x + 2} = ?$



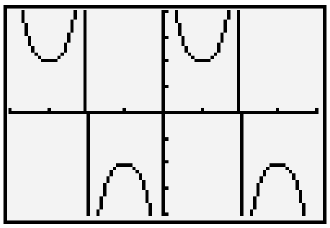
$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-4, 4, 1]$

65. $\dfrac{1}{\sec x + \tan x} + \dfrac{1}{\sec x - \tan x} = ?$



$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-4, 4, 1]$

66. $\dfrac{1 + \cos x}{\sin x} + \dfrac{\sin x}{1 + \cos x} = ?$



$[-2\pi, 2\pi, \frac{\pi}{2}]$ by $[-4, 4, 1]$

In Exercises 67–74, rewrite each expression in terms of the given function or functions.

67. $\dfrac{\tan x + \cot x}{\csc x}$; $\cos x$

68. $\dfrac{\sec x + \csc x}{1 + \tan x}$; $\sin x$

69. $\dfrac{\cos x}{1 + \sin x} + \tan x$; $\cos x$

70. $\dfrac{1}{\sin x \cos x} - \cot x$; $\cot x$

71. $\dfrac{1}{1 - \cos x} - \dfrac{\cos x}{1 + \cos x}$; $\csc x$

72. $(\sec x + \csc x)(\sin x + \cos x) - 2 - \cot x$; $\tan x$

73. $\dfrac{1}{\csc x - \sin x}$; $\sec x$ and $\tan x$

74. $\dfrac{1 - \sin x}{1 + \sin x} - \dfrac{1 + \sin x}{1 - \sin x}$; $\sec x$ and $\tan x$

Writing in Mathematics

75. Explain how to verify an identity.

76. Describe two strategies that can be used to verify identities.

77. Describe how you feel when you successfully verify a difficult identity. What other activities do you engage in that evoke the same feelings?

78. A 10-point question on a quiz asks students to verify the identity

$$\frac{\sin^2 x - \cos^2 x}{\sin x + \cos x} = \sin x - \cos x.$$

One student begins with the left side and obtains the right side as follows:

$$\frac{\sin^2 x - \cos^2 x}{\sin x + \cos x} = \frac{\sin^2 x}{\sin x} - \frac{\cos^2 x}{\cos x} = \sin x - \cos x.$$

How many points (out of 10) would you give this student? Explain your answer.

Technology Exercises

In Exercises 79–87, graph each side of the equation in the same viewing rectangle. If the graphs appear to coincide, verify that the equation is an identity. If the graphs do not appear to coincide, this indicates the equation is not an identity. In these exercises, find a value of x for which both sides are defined but not equal.

79. $\tan x = \sec x(\sin x - \cos x) + 1$

80. $\sin x = -\cos x \tan(-x)$

81. $\sin\left(x + \dfrac{\pi}{4}\right) = \sin x + \sin\dfrac{\pi}{4}$

82. $\cos\left(x + \dfrac{\pi}{4}\right) = \cos x + \cos\dfrac{\pi}{4}$

83. $\cos(x + \pi) = \cos x$

84. $\sin(x + \pi) = \sin x$

85. $\dfrac{\sin x}{1 - \cos^2 x} = \csc x$

86. $\sin x - \sin x \cos^2 x = \sin^3 x$

87. $\sqrt{\sin^2 x + \cos^2 x} = \sin x + \cos x$

Critical Thinking Exercises

Make Sense? *In Exercises 88–91, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

88. The word *identity* is used in different ways in additive identity, multiplicative identity, and trigonometric identity.

89. To prove a trigonometric identity, I select one side of the equation and transform it until it is the other side of the equation, or I manipulate both sides to a common trigonometric expression.

90. In order to simplify $\frac{\cos x}{1 - \sin x} - \frac{\sin x}{\cos x}$, I need to know how to subtract rational expressions with unlike denominators.

91. The most efficient way that I can simplify $\frac{(\sec x + 1)(\sec x - 1)}{\sin^2 x}$ is to immediately rewrite the expression in terms of cosines and sines.

In Exercises 92–95, verify each identity.

92. $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \sin x \cos x$

93. $\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \frac{\sin x + 1}{\cos x}$

94. $\ln|\sec x| = -\ln|\cos x|$

95. $\ln e^{\tan^2 x - \sec^2 x} = -1$

96. Use one of the fundamental identities in the box on page 586 to create an original identity.

Group Exercise

97. Group members are to write a helpful list of items for a pamphlet called "The Underground Guide to Verifying Identities." The pamphlet will be used primarily by students who sit, stare, and freak out every time they are asked to verify an identity. List easy ways to remember the fundamental identities. What helpful guidelines can you offer from the perspective of a student that you probably won't find in math books? If you have your own strategies that work particularly well, include them in the pamphlet.

Preview Exercises

Exercises 98–100 will help you prepare for the material covered in the next section.

98. Give exact values for cos 30°, sin 30°, cos 60°, sin 60°, cos 90°, and sin 90°.

99. Use the appropriate values from Exercise 98 to answer each of the following.

a. Is $\cos(30° + 60°)$, or cos 90°, equal to $\cos 30° + \cos 60°$?

b. Is $\cos(30° + 60°)$, or cos 90°, equal to $\cos 30° \cos 60° - \sin 30° \sin 60°$?

100. Use the appropriate values from Exercise 98 to answer each of the following.

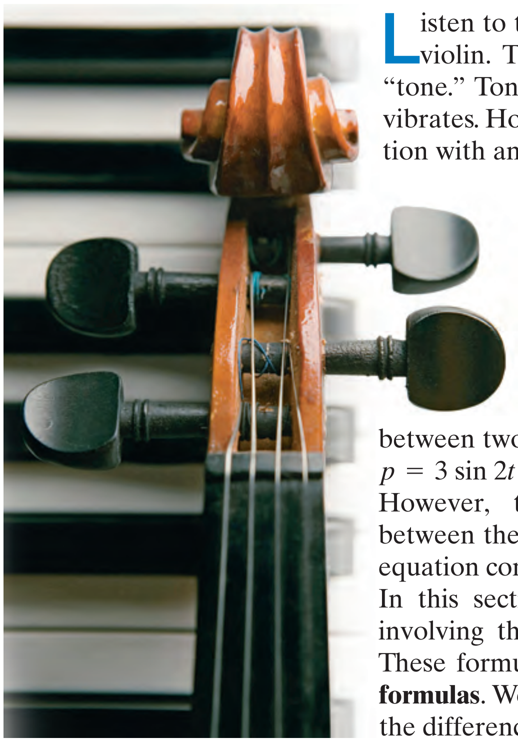
a. Is $\sin(30° + 60°)$, or sin 90°, equal to $\sin 30° + \sin 60°$?

b. Is $\sin(30° + 60°)$, or sin 90°, equal to $\sin 30° \cos 60° + \cos 30° \sin 60°$?

Section 5.2 Sum and Difference Formulas

Objectives

1. Use the formula for the cosine of the difference of two angles.
2. Use sum and difference formulas for cosines and sines.
3. Use sum and difference formulas for tangents.



Listen to the same note played on a piano and a violin. The notes have a different quality or "tone." Tone depends on the way an instrument vibrates. However, the less than 1% of the population with amusia, or true tone deafness, cannot tell the two sounds apart. Even simple, familiar tunes such as *Happy Birthday* and *Jingle Bells* are mystifying to amusics.

When a note is played, it vibrates at a specific fundamental frequency and has a particular amplitude. Amusics cannot tell the difference between two sounds from tuning forks modeled by $p = 3 \sin 2t$ and $p = 2 \sin(2t + \pi)$, respectively. However, they can recognize the difference between the two equations. Notice that the second equation contains the sine of the sum of two angles. In this section, we will be developing identities involving the sums or differences of two angles. These formulas are called the **sum and difference formulas**. We begin with $\cos(\alpha - \beta)$, the cosine of the difference of two angles.