Critical Thinking Exercises

Make Sense? *In Exercises 88–91, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

88. The word *identity* is used in different ways in additive identity, multiplicative identity, and trigonometric identity.

89. To prove a trigonometric identity, I select one side of the equation and transform it until it is the other side of the equation, or I manipulate both sides to a common trigonometric expression.

90. In order to simplify $\frac{\cos x}{1 - \sin x} - \frac{\sin x}{\cos x}$, I need to know how to subtract rational expressions with unlike denominators.

91. The most efficient way that I can simplify $\frac{(\sec x + 1)(\sec x - 1)}{\sin^2 x}$ is to immediately rewrite the expression in terms of cosines and sines.

In Exercises 92–95, verify each identity.

92. $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \sin x \cos x$

93. $\frac{\sin x - \cos x + 1}{\sin x + \cos x - 1} = \frac{\sin x + 1}{\cos x}$

94. $\ln|\sec x| = -\ln|\cos x|$

95. $\ln e^{\tan^2 x - \sec^2 x} = -1$

96. Use one of the fundamental identities in the box on page 586 to create an original identity.

Group Exercise

97. Group members are to write a helpful list of items for a pamphlet called "The Underground Guide to Verifying Identities." The pamphlet will be used primarily by students who sit, stare, and freak out every time they are asked to verify an identity. List easy ways to remember the fundamental identities. What helpful guidelines can you offer from the perspective of a student that you probably won't find in math books? If you have your own strategies that work particularly well, include them in the pamphlet.

Preview Exercises

Exercises 98–100 will help you prepare for the material covered in the next section.

98. Give exact values for $\cos 30°$, $\sin 30°$, $\cos 60°$, $\sin 60°$, $\cos 90°$, and $\sin 90°$.

99. Use the appropriate values from Exercise 98 to answer each of the following.

a. Is $\cos(30° + 60°)$, or $\cos 90°$, equal to $\cos 30° + \cos 60°$?

b. Is $\cos(30° + 60°)$, or $\cos 90°$, equal to $\cos 30° \cos 60° - \sin 30° \sin 60°$?

100. Use the appropriate values from Exercise 98 to answer each of the following.

a. Is $\sin(30° + 60°)$, or $\sin 90°$, equal to $\sin 30° + \sin 60°$?

b. Is $\sin(30° + 60°)$, or $\sin 90°$, equal to $\sin 30° \cos 60° + \cos 30° \sin 60°$?

Section 5.2 Sum and Difference Formulas

Objectives

1. Use the formula for the cosine of the difference of two angles.
2. Use sum and difference formulas for cosines and sines.
3. Use sum and difference formulas for tangents.

Listen to the same note played on a piano and a violin. The notes have a different quality or "tone." Tone depends on the way an instrument vibrates. However, the less than 1% of the population with amusia, or true tone deafness, cannot tell the two sounds apart. Even simple, familiar tunes such as *Happy Birthday* and *Jingle Bells* are mystifying to amusics.

When a note is played, it vibrates at a specific fundamental frequency and has a particular amplitude. Amusics cannot tell the difference between two sounds from tuning forks modeled by $p = 3 \sin 2t$ and $p = 2\sin(2t + \pi)$, respectively. However, they can recognize the difference between the two equations. Notice that the second equation contains the sine of the sum of two angles. In this section, we will be developing identities involving the sums or differences of two angles. These formulas are called the **sum and difference formulas**. We begin with $\cos(\alpha - \beta)$, the cosine of the difference of two angles.

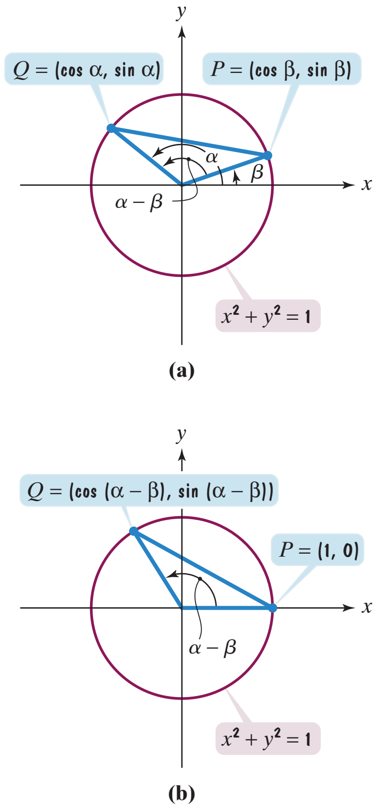


Figure 5.1 Using the unit circle and PQ to develop a formula for $\cos(\alpha - \beta)$

The Cosine of the Difference of Two Angles

The Cosine of the Difference of Two Angles

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

The cosine of the difference of two angles equals the cosine of the first angle times the cosine of the second angle plus the sine of the first angle times the sine of the second angle.

We use **Figure 5.1** to prove the identity in the box. The graph in **Figure 5.1(a)** shows a unit circle, $x^2 + y^2 = 1$. The figure uses the definitions of the cosine and sine functions as the x- and y-coordinates of points along the unit circle. For example, point P corresponds to angle β. By definition, the x-coordinate of P is $\cos\beta$ and the y-coordinate is $\sin\beta$. Similarly, point Q corresponds to angle α. By definition, the x-coordinate of Q is $\cos\alpha$ and the y-coordinate is $\sin\alpha$.

Note that if we draw a line segment between points P and Q, a triangle is formed. Angle $\alpha - \beta$ is one of the angles of this triangle. What happens if we rotate this triangle so that point P falls on the x-axis at $(1, 0)$? The result is shown in **Figure 5.1(b)**. This rotation changes the coordinates of points P and Q. However, it has no effect on the length of line segment PQ.

We can use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to find an expression for PQ in **Figure 5.1(a)** and in **Figure 5.1(b)**. By equating the two expressions for PQ, we will obtain the identity for the cosine of the difference of two angles, $\alpha - \beta$. We first apply the distance formula in **Figure 5.1(a)**.

$$PQ = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$

Apply the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to find the distance between $(\cos\beta, \sin\beta)$ and $(\cos\alpha, \sin\alpha)$.

$$= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta}$$

Square each expression using $(A - B)^2 = A^2 - 2AB + B^2$.

$$= \sqrt{(\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

Regroup terms to apply a Pythagorean identity.

$$= \sqrt{1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

Because $\sin^2 x + \cos^2 x = 1$, each expression in parentheses equals 1.

$$= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

Simplify.

Next, we apply the distance formula in **Figure 5.1(b)** to obtain a second expression for PQ. We let $(x_1, y_1) = (1, 0)$ and $(x_2, y_2) = (\cos(\alpha - \beta), \sin(\alpha - \beta))$.

$$PQ = \sqrt{[\cos(\alpha - \beta) - 1]^2 + [\sin(\alpha - \beta) - 0]^2}$$

Apply the distance formula to find the distance between $(1, 0)$ and $(\cos(\alpha - \beta), \sin(\alpha - \beta))$.

$$= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)}$$

Square each expression.

$$= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)}$$

Using a Pythagorean identity, $\sin^2(\alpha - \beta) + \cos^2(\alpha - \beta) = 1$.

$$= \sqrt{1 - 2\cos(\alpha - \beta) + 1}$$

Use a Pythagorean identity.

$$= \sqrt{2 - 2\cos(\alpha - \beta)}$$

Simplify.

Now we equate the two expressions for PQ.

$$\sqrt{2 - 2\cos(\alpha - \beta)} = \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$ The rotation does not change the length of PQ.

$$2 - 2\cos(\alpha - \beta) = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$ Square both sides to eliminate radicals.

$$-2\cos(\alpha - \beta) = -2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$ Subtract 2 from both sides of the equation.

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$ Divide both sides of the equation by -2.

1 Use the formula for the cosine of the difference of two angles.

This proves the identity for the cosine of the difference of two angles.

Now that we see where the identity for the cosine of the difference of two angles comes from, let's look at some applications of this result.

EXAMPLE 1 Using the Difference Formula for Cosines to Find an Exact Value

Find the exact value of $\cos 15°$.

Solution We know exact values for trigonometric functions of 60° and 45°. Thus, we write 15° as $60° - 45°$ and use the difference formula for cosines.

$$\cos 15° = \cos(60° - 45°)$$

$$= \cos 60° \cos 45° + \sin 60° \sin 45°$$ $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$ Substitute exact values from memory or use special right triangles.

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$ Multiply.

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$ Add.

Check Point 1 We know that $\cos 30° = \frac{\sqrt{3}}{2}$. Obtain this exact value using $\cos 30° = \cos(90° - 60°)$ and the difference formula for cosines.

EXAMPLE 2 Using the Difference Formula for Cosines to Find an Exact Value

Find the exact value of $\cos 80° \cos 20° + \sin 80° \sin 20°$.

Solution The given expression is the right side of the formula for $\cos(\alpha - \beta)$ with $\alpha = 80°$ and $\beta = 20°$.

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\cos 80° \cos 20° + \sin 80° \sin 20° = \cos(80° - 20°) = \cos 60° = \frac{1}{2}$$

Check Point 2 Find the exact value of

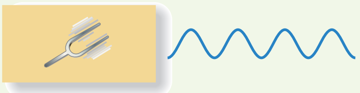
$$\cos 70° \cos 40° + \sin 70° \sin 40°.$$

Sound Quality and Amusia

People with true tone deafness cannot hear the difference among tones produced by a tuning fork, a flute, an oboe, and a violin. They cannot dance or tell the difference between harmony and dissonance. People with amusia appear to have been born without the wiring necessary to process music. Intriguingly, they show no overt signs of brain damage and their brain scans appear normal. Thus, they can visually recognize the difference among sound waves that produce varying sound qualities.

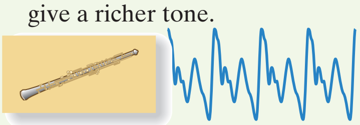
Varying Sound Qualities

- Tuning fork: Sound waves are rounded and regular, giving a pure and gentle tone.



- Flute: Sound waves are smooth and give a fluid tone.

- Oboe: Rapid wave changes give a richer tone.



- Violin: Jagged waves give a brighter harsher tone.

EXAMPLE 3 Verifying an Identity

Verify the identity: $\dfrac{\cos(\alpha - \beta)}{\sin \alpha \cos \beta} = \cot \alpha + \tan \beta$.

Solution We work with the left side.

$$\frac{\cos(\alpha - \beta)}{\sin \alpha \cos \beta} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\sin \alpha \cos \beta}$$ Use the formula for $\cos(\alpha - \beta)$.

$$= \frac{\cos \alpha \cos \beta}{\sin \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\sin \alpha \cos \beta}$$ Divide each term in the numerator by $\sin \alpha \cos \beta$.

$$= \frac{\cos \alpha}{\sin \alpha} \cdot \frac{\cos \beta}{\cos \beta} + \frac{\sin \alpha}{\sin \alpha} \cdot \frac{\sin \beta}{\cos \beta}$$ This step can be done mentally. We wanted you to see the substitutions that follow.

$$= \cot \alpha \cdot 1 + 1 \cdot \tan \beta$$ Use quotient identities.

$$= \cot \alpha + \tan \beta$$ Simplify.

We worked with the left side and arrived at the right side. Thus, the identity is verified.

Check Point 3 Verify the identity: $\dfrac{\cos(\alpha - \beta)}{\cos \alpha \cos \beta} = 1 + \tan \alpha \tan \beta$.

Technology

Graphic Connections

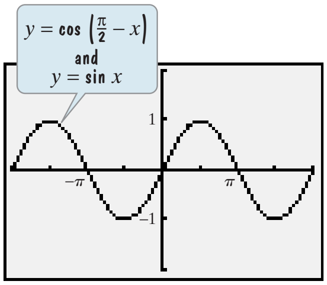
The graphs of

$$y = \cos\left(\frac{\pi}{2} - x\right)$$

and

$$y = \sin x$$

are shown in the same viewing rectangle. The graphs are the same. The displayed math on the right with the voice balloon on top shows the equivalence algebraically.



The difference formula for cosines is used to establish other identities. For example, in our work with right triangles, we noted that cofunctions of complements are equal. Thus, because $\dfrac{\pi}{2} - \theta$ and θ are complements,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta.$$

We can use the formula for $\cos(\alpha - \beta)$ to prove this cofunction identity for all angles.

Apply $\cos(\alpha - \beta)$ with $\alpha = \frac{\pi}{2}$ and $\beta = \theta$.
$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos \frac{\pi}{2} \cos \theta + \sin \frac{\pi}{2} \sin \theta$$
$$= 0 \cdot \cos \theta + 1 \cdot \sin \theta$$
$$= \sin \theta$$

2 Use sum and difference formulas for cosines and sines.

Sum and Difference Formulas for Cosines and Sines

Our formula for $\cos(\alpha - \beta)$ can be used to verify an identity for a sum involving cosines, as well as identities for a sum and a difference for sines.

Sum and Difference Formulas for Cosines and Sines

1. $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
2. $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
3. $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
4. $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

Up to now, we have concentrated on the second formula in the box on the previous page, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. The first identity gives a formula for the cosine of the sum of two angles. It is proved as follows:

$$\cos(\alpha + \beta) = \cos[\alpha - (-\beta)]$$ Express addition as subtraction of an additive inverse.

$$= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$ Use the difference formula for cosines.

$$= \cos\alpha\cos\beta + \sin\alpha(-\sin\beta)$$ Cosine is even: $\cos(-\beta) = \cos\beta$. Sine is odd: $\sin(-\beta) = -\sin\beta$.

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$ Simplify.

Thus, the cosine of the sum of two angles equals the cosine of the first angle times the cosine of the second angle minus the sine of the first angle times the sine of the second angle.

The third identity in the box gives a formula for $\sin(\alpha + \beta)$, the sine of the sum of two angles. It is proved as follows:

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right]$$ Use a cofunction identity: $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$.

$$= \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$$ Regroup.

$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$$ Use the difference formula for cosines.

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$ Use cofunction identities.

Thus, the sine of the sum of two angles equals the sine of the first angle times the cosine of the second angle plus the cosine of the first angle times the sine of the second angle.

The final identity in the box, $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$, gives a formula for $\sin(\alpha - \beta)$, the sine of the difference of two angles. It is proved by writing $\sin(\alpha - \beta)$ as $\sin[\alpha + (-\beta)]$ and then using the formula for the sine of a sum.

EXAMPLE 4 Using the Sine of a Sum to Find an Exact Value

Find the exact value of $\sin\frac{7\pi}{12}$ using the fact that $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$.

Solution We apply the formula for the sine of a sum.

$$\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$ $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$ Substitute exact values.

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$ Simplify.

Check Point 4 Find the exact value of $\sin\frac{5\pi}{12}$ using the fact that

$$\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}.$$

EXAMPLE 5 Finding Exact Values

Suppose that $\sin\alpha = \frac{12}{13}$ for a quadrant II angle α and $\sin\beta = \frac{3}{5}$ for a quadrant I angle β. Find the exact value of each of the following:

a. $\cos\alpha$ **b.** $\cos\beta$ **c.** $\cos(\alpha+\beta)$ **d.** $\sin(\alpha+\beta)$.

Solution

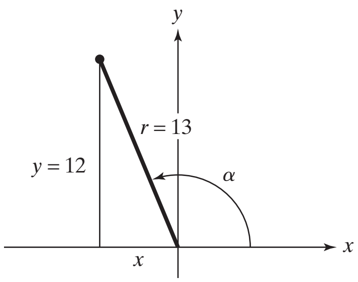


Figure 5.2 $\sin\alpha = \frac{12}{13}$: α lies in quadrant II.

a. We find $\cos\alpha$ using a sketch that illustrates

$$\sin\alpha = \frac{12}{13} = \frac{y}{r}.$$

Figure 5.2 shows a quadrant II angle α with $\sin\alpha = \frac{12}{13}$. We find x using $x^2 + y^2 = r^2$. Because α lies in quadrant II, x is negative.

$x^2 + 12^2 = 13^2$	$x^2 + y^2 = r^2$
$x^2 + 144 = 169$	Square 12 and 13, respectively.
$x^2 = 25$	Subtract 144 from both sides.
$x = -\sqrt{25} = -5$	If $x^2 = 25$, then $x = \pm\sqrt{25} = \pm 5$. Choose $x = -\sqrt{25}$ because in quadrant II, x is negative.

Thus,

$$\cos\alpha = \frac{x}{r} = \frac{-5}{13} = -\frac{5}{13}.$$

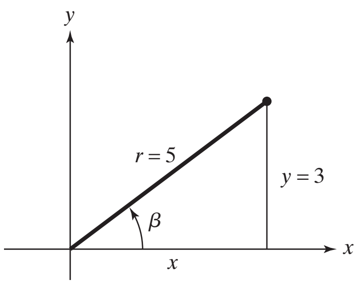


Figure 5.3 $\sin\beta = \frac{3}{5}$: β lies in quadrant I.

b. We find $\cos\beta$ using a sketch that illustrates

$$\sin\beta = \frac{3}{5} = \frac{y}{r}.$$

Figure 5.3 shows a quadrant I angle β with $\sin\beta = \frac{3}{5}$. We find x using $x^2 + y^2 = r^2$.

$x^2 + 3^2 = 5^2$	$x^2 + y^2 = r^2$
$x^2 + 9 = 25$	Square 3 and 5, respectively.
$x^2 = 16$	Subtract 9 from both sides.
$x = \sqrt{16} = 4$	If $x^2 = 16$, then $x = \pm\sqrt{16} = \pm 4$. Choose $x = \sqrt{16}$ because in quadrant 1, x is positive.

Thus,

$$\cos\beta = \frac{x}{r} = \frac{4}{5}.$$

We use the given values and the exact values that we determined to find exact values for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

These values are given. These are the values we found.

$$\underbrace{\sin\alpha = \frac{12}{13},\ \sin\beta = \frac{3}{5}}_{\text{These values are given.}} \qquad \underbrace{\cos\alpha = -\frac{5}{13},\ \cos\beta = \frac{4}{5}}_{\text{These are the values we found.}}$$

c. We use the formula for the cosine of a sum.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= \left(-\frac{5}{13}\right)\left(\frac{4}{5}\right) - \frac{12}{13}\left(\frac{3}{5}\right) = -\frac{56}{65}$$

d. We use the formula for the sine of a sum.

These values are given.

$\sin\alpha = \frac{12}{13}$, $\sin\beta = \frac{3}{5}$

These are the values we found.

$\cos\alpha = -\frac{5}{13}$, $\cos\beta = \frac{4}{5}$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \frac{12}{13}\cdot\frac{4}{5} + \left(-\frac{5}{13}\right)\cdot\frac{3}{5} = \frac{33}{65}$$

Check Point 5 Suppose that $\sin\alpha = \frac{4}{5}$ for a quadrant II angle α and $\sin\beta = \frac{1}{2}$ for a quadrant I angle β. Find the exact value of each of the following:

a. $\cos\alpha$ **b.** $\cos\beta$ **c.** $\cos(\alpha+\beta)$ **d.** $\sin(\alpha+\beta)$.

EXAMPLE 6 Verifying Observations on a Graphing Utility

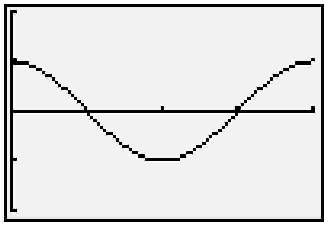


Figure 5.4 The graph of $y = \sin\left(x - \frac{3\pi}{2}\right)$ in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$ viewing rectangle

Figure 5.4 shows the graph of $y = \sin\left(x - \frac{3\pi}{2}\right)$ in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$ viewing rectangle.

a. Describe the graph using another equation.

b. Verify that the two equations are equivalent.

Solution

a. The graph appears to be the cosine curve $y = \cos x$. It cycles through maximum, intercept, minimum, intercept, and back to maximum. Thus, $y = \cos x$ also describes the graph.

b. We must show that

$$\sin\left(x - \frac{3\pi}{2}\right) = \cos x.$$

We apply the formula for the sine of a difference on the left side.

$$\sin\left(x - \frac{3\pi}{2}\right) = \sin x\cos\frac{3\pi}{2} - \cos x\sin\frac{3\pi}{2}$$ $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \sin x\cdot 0 - \cos x(-1)$$ $\cos\frac{3\pi}{2} = 0$ and $\sin\frac{3\pi}{2} = -1$

$$= \cos x$$ Simplify.

This verifies our observation that $y = \sin\left(x - \frac{3\pi}{2}\right)$ and $y = \cos x$ describe the same graph.

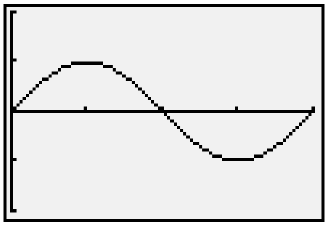


Figure 5.5

Check Point 6 **Figure 5.5** shows the graph of $y = \cos\left(x + \frac{3\pi}{2}\right)$ in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$ viewing rectangle.

a. Describe the graph using another equation.

b. Verify that the two equations are equivalent.

3 Use sum and difference formulas for tangents.

Sum and Difference Formulas for Tangents

By writing $\tan(\alpha+\beta)$ as the quotient of $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, we can develop a formula for the tangent of a sum. Writing subtraction as addition of an inverse leads to a formula for the tangent of a difference.

Discovery

Derive the sum and difference formulas for tangents by working Exercises 55 and 56 in Exercise Set 5.2.

Sum and Difference Formulas for Tangents

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

The tangent of the sum of two angles equals the tangent of the first angle plus the tangent of the second angle divided by 1 minus their product.

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

The tangent of the difference of two angles equals the tangent of the first angle minus the tangent of the second angle divided by 1 plus their product.

EXAMPLE 7 Verifying an Identity

Verify the identity: $\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$.

Solution We work with the left side.

$$\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - \tan\frac{\pi}{4}}{1 + \tan x \tan\frac{\pi}{4}} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{\tan x - 1}{1 + \tan x \cdot 1} \qquad \tan\frac{\pi}{4} = 1$$

$$= \frac{\tan x - 1}{\tan x + 1}$$

Check Point 7 Verify the identity: $\tan(x + \pi) = \tan x$.

Exercise Set 5.2

Practice Exercises

Use the formula for the cosine of the difference of two angles to solve Exercises 1–12.

In Exercises 1–4, find the exact value of each expression.

1. $\cos(45° - 30°)$
2. $\cos(120° - 45°)$
3. $\cos\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$
4. $\cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)$

In Exercises 5–8, each expression is the right side of the formula for $\cos(\alpha - \beta)$ with particular values for α and β.

a. *Identify α and β in each expression.*

b. *Write the expression as the cosine of an angle.*

c. *Find the exact value of the expression.*

5. $\cos 50° \cos 20° + \sin 50° \sin 20°$
6. $\cos 50° \cos 5° + \sin 50° \sin 5°$
7. $\cos\frac{5\pi}{12}\cos\frac{\pi}{12} + \sin\frac{5\pi}{12}\sin\frac{\pi}{12}$
8. $\cos\frac{5\pi}{18}\cos\frac{\pi}{9} + \sin\frac{5\pi}{18}\sin\frac{\pi}{9}$

In Exercises 9–12, verify each identity.

9. $\frac{\cos(\alpha - \beta)}{\cos\alpha \sin\beta} = \tan\alpha + \cot\beta$
10. $\frac{\cos(\alpha - \beta)}{\sin\alpha \sin\beta} = \cot\alpha\cot\beta + 1$
11. $\cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos x + \sin x)$
12. $\cos\left(x - \frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}(\cos x + \sin x)$

Use one or more of the six sum and difference identities to solve Exercises 13–54.

In Exercises 13–24, find the exact value of each expression.

13. $\sin(45° - 30°)$

14. $\sin(60° - 45°)$

15. $\sin 105°$

16. $\sin 75°$

17. $\cos(135° + 30°)$

18. $\cos(240° + 45°)$

19. $\cos 75°$

20. $\cos 105°$

21. $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

22. $\tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

23. $\tan\left(\frac{4\pi}{3} - \frac{\pi}{4}\right)$

24. $\tan\left(\frac{5\pi}{3} - \frac{\pi}{4}\right)$

In Exercises 25–32, write each expression as the sine, cosine, or tangent of an angle. Then find the exact value of the expression.

25. $\sin 25° \cos 5° + \cos 25° \sin 5°$

26. $\sin 40° \cos 20° + \cos 40° \sin 20°$

27. $\dfrac{\tan 10° + \tan 35°}{1 - \tan 10° \tan 35°}$

28. $\dfrac{\tan 50° - \tan 20°}{1 + \tan 50° \tan 20°}$

29. $\sin\frac{5\pi}{12}\cos\frac{\pi}{4} - \cos\frac{5\pi}{12}\sin\frac{\pi}{4}$

30. $\sin\frac{7\pi}{12}\cos\frac{\pi}{12} - \cos\frac{7\pi}{12}\sin\frac{\pi}{12}$

31. $\dfrac{\tan\frac{\pi}{5} - \tan\frac{\pi}{30}}{1 + \tan\frac{\pi}{5}\tan\frac{\pi}{30}}$

32. $\dfrac{\tan\frac{\pi}{5} + \tan\frac{4\pi}{5}}{1 - \tan\frac{\pi}{5}\tan\frac{4\pi}{5}}$

In Exercises 33–54, verify each identity.

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

34. $\sin\left(x + \frac{3\pi}{2}\right) = -\cos x$

35. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

36. $\cos(\pi - x) = -\cos x$

37. $\tan(2\pi - x) = -\tan x$

38. $\tan(\pi - x) = -\tan x$

39. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin\alpha \cos\beta$

40. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos\alpha \cos\beta$

41. $\dfrac{\sin(\alpha - \beta)}{\cos\alpha\cos\beta} = \tan\alpha - \tan\beta$

42. $\dfrac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

43. $\tan\left(\theta + \frac{\pi}{4}\right) = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

44. $\tan\left(\frac{\pi}{4} - \theta\right) = \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

45. $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\beta - \sin^2\alpha$

46. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \cos^2\beta - \cos^2\alpha$

47. $\dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta}$

48. $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1 - \tan\alpha\tan\beta}{1 + \tan\alpha\tan\beta}$

49. $\dfrac{\cos(x + h) - \cos x}{h} = \cos x\dfrac{\cos h - 1}{h} - \sin x\dfrac{\sin h}{h}$

50. $\dfrac{\sin(x + h) - \sin x}{h} = \cos x\dfrac{\sin h}{h} + \sin x\dfrac{\cos h - 1}{h}$

51. $\sin 2\alpha = 2\sin\alpha\cos\alpha$

Hint: Write $\sin 2\alpha$ as $\sin(\alpha + \alpha)$.

52. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

Hint: Write $\cos 2\alpha$ as $\cos(\alpha + \alpha)$.

53. $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

Hint: Write $\tan 2\alpha$ as $\tan(\alpha + \alpha)$.

54. $\tan\left(\frac{\pi}{4} + \alpha\right) - \tan\left(\frac{\pi}{4} - \alpha\right) = 2\tan 2\alpha$

Hint: Use the result in Exercise 53.

55. Derive the identity for $\tan(\alpha + \beta)$ using

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}.$$

After applying the formulas for sums of sines and cosines, divide the numerator and denominator by $\cos\alpha\cos\beta$.

56. Derive the identity for $\tan(\alpha - \beta)$ using

$$\tan(\alpha - \beta) = \tan[\alpha + (-\beta)].$$

After applying the formula for the tangent of the sum of two angles, use the fact that the tangent is an odd function.

In Exercises 57–64, find the exact value of the following under the given conditions:

a. $\cos(\alpha + \beta)$ **b.** $\sin(\alpha + \beta)$ **c.** $\tan(\alpha + \beta)$.

57. $\sin\alpha = \frac{3}{5}$, α lies in quadrant I, and $\sin\beta = \frac{5}{13}$, β lies in quadrant II.

58. $\sin\alpha = \frac{4}{5}$, α lies in quadrant I, and $\sin\beta = \frac{7}{25}$, β lies in quadrant II.

59. $\tan\alpha = -\frac{3}{4}$, α lies in quadrant II, and $\cos\beta = \frac{1}{3}$, β lies in quadrant I.

60. $\tan\alpha = -\frac{4}{3}$, α lies in quadrant II, and $\cos\beta = \frac{2}{3}$, β lies in quadrant I.

61. $\cos\alpha = \frac{8}{17}$, α lies in quadrant IV, and $\sin\beta = -\frac{1}{2}$, β lies in quadrant III.

62. $\cos\alpha = \frac{1}{2}$, α lies in quadrant IV, and $\sin\beta = -\frac{1}{3}$, β lies in quadrant III.

63. $\tan\alpha = \frac{3}{4}$, $\pi < \alpha < \frac{3\pi}{2}$, and $\cos\beta = \frac{1}{4}$, $\frac{3\pi}{2} < \beta < 2\pi$.

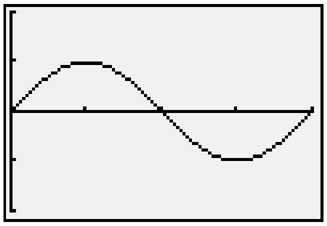
64. $\sin\alpha = \frac{5}{6}$, $\frac{\pi}{2} < \alpha < \pi$, and $\tan\beta = \frac{3}{7}$, $\pi < \beta < \frac{3\pi}{2}$.

In Exercises 65–68, the graph with the given equation is shown in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ *by* $[-2, 2, 1]$ *viewing rectangle.*

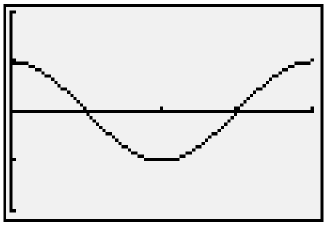
a. *Describe the graph using another equation.*

b. *Verify that the two equations are equivalent.*

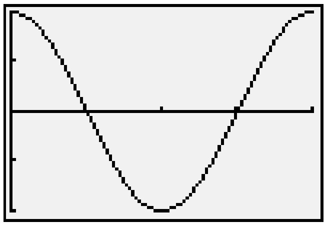
65. $y = \sin(\pi - x)$



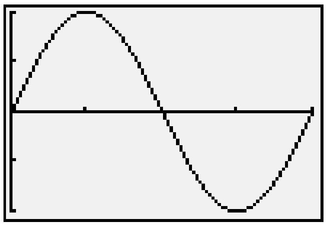
66. $y = \cos(x - 2\pi)$



67. $y = \sin\left(x + \frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2} - x\right)$



68. $y = \cos\left(x - \frac{\pi}{2}\right) - \cos\left(x + \frac{\pi}{2}\right)$



Practice Plus

In Exercises 69–74, rewrite each expression as a simplified expression containing one term.

69. $\cos(\alpha + \beta)\cos\beta + \sin(\alpha + \beta)\sin\beta$

70. $\sin(\alpha - \beta)\cos\beta + \cos(\alpha - \beta)\sin\beta$

71. $\dfrac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{\cos(\alpha + \beta) + \cos(\alpha - \beta)}$

72. $\dfrac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{-\sin(\alpha - \beta) + \sin(\alpha + \beta)}$

73. $\cos\left(\frac{\pi}{6} + \alpha\right)\cos\left(\frac{\pi}{6} - \alpha\right) - \sin\left(\frac{\pi}{6} + \alpha\right)\sin\left(\frac{\pi}{6} - \alpha\right)$

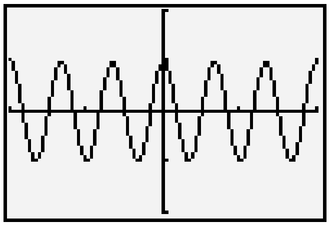
(Do not use four different identities to solve this exercise.)

74. $\sin\left(\frac{\pi}{3} - \alpha\right)\cos\left(\frac{\pi}{3} + \alpha\right) + \cos\left(\frac{\pi}{3} - \alpha\right)\sin\left(\frac{\pi}{3} + \alpha\right)$

(Do not use four different identities to solve this exercise.)

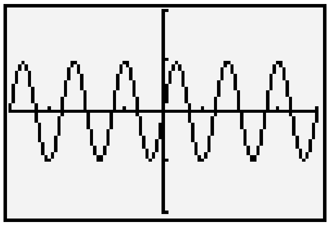
In Exercises 75–78, half of an identity and the graph of this half are given. Use the graph to make a conjecture as to what the right side of the identity should be. Then prove your conjecture.

75. $\cos 2x \cos 5x + \sin 2x \sin 5x = ?$



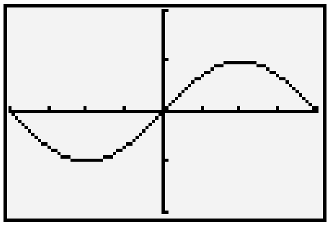
$\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$

76. $\sin 5x \cos 2x - \cos 5x \sin 2x = ?$



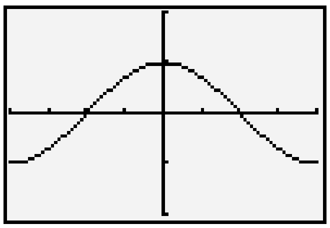
$\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$

77. $\sin\frac{5x}{2}\cos 2x - \cos\frac{5x}{2}\sin 2x = ?$



$\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$

78. $\cos\frac{5x}{2}\cos 2x + \sin\frac{5x}{2}\sin 2x = ?$



$\left[-2\pi, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$

Application Exercises

79. A ball attached to a spring is raised 2 feet and released with an initial vertical velocity of 3 feet per second. The distance of the ball from its rest position after t seconds is given by $d = 2 \cos t + 3 \sin t$. Show that

$$2 \cos t + 3 \sin t = \sqrt{13} \cos(t - \theta),$$

where θ lies in quadrant I and $\tan \theta = \frac{3}{2}$. Use the identity to find the amplitude and the period of the ball's motion.

80. A tuning fork is held a certain distance from your ears and struck. Your eardrums' vibrations after t seconds are given by $p = 3 \sin 2t$. When a second tuning fork is struck, the formula $p = 2 \sin(2t + \pi)$ describes the effects of the sound on the eardrums' vibrations. The total vibrations are given by $p = 3 \sin 2t + 2 \sin(2t + \pi)$.

a. Simplify p to a single term containing the sine.

b. If the amplitude of p is zero, no sound is heard. Based on your equation in part (a), does this occur with the two tuning forks in this exercise? Explain your answer.

Writing in Mathematics

In Exercises 81–86, use words to describe the formula for each of the following:

81. the cosine of the difference of two angles.

82. the cosine of the sum of two angles.

83. the sine of the sum of two angles.

84. the sine of the difference of two angles.

85. the tangent of the difference of two angles.

86. the tangent of the sum of two angles.

87. The distance formula and the definitions for cosine and sine are used to prove the formula for the cosine of the difference of two angles. This formula logically leads the way to the other sum and difference identities. Using this development of ideas and formulas, describe a characteristic of mathematical logic.

Technology Exercises

In Exercises 88–93, graph each side of the equation in the same viewing rectangle. If the graphs appear to coincide, verify that the equation is an identity. If the graphs do not appear to coincide, this indicates that the equation is not an identity. In these exercises, find a value of x for which both sides are defined but not equal.

88. $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$

89. $\tan(\pi - x) = -\tan x$

90. $\sin\left(x + \frac{\pi}{2}\right) = \sin x + \sin \frac{\pi}{2}$

91. $\cos\left(x + \frac{\pi}{2}\right) = \cos x + \cos \frac{\pi}{2}$

92. $\cos 1.2x \cos 0.8x - \sin 1.2x \sin 0.8x = \cos 2x$

93. $\sin 1.2x \cos 0.8x + \cos 1.2x \sin 0.8x = \sin 2x$

Critical Thinking Exercises

Make Sense? *In Exercises 94–97, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

94. I've noticed that for sine, cosine, and tangent, the trig function for the sum of two angles is not equal to that trig function of the first angle plus that trig function of the second angle.

95. After using an identity to determine the exact value of sin 105°, I verified the result with a calculator.

96. Using sum and difference formulas, I can find exact values for sine, cosine, and tangent at any angle.

97. After the difference formula for cosines is verified, I noticed that the other sum and difference formulas are verified relatively quickly.

98. Verify the identity:

$$\frac{\sin(x - y)}{\cos x \cos y} + \frac{\sin(y - z)}{\cos y \cos z} + \frac{\sin(z - x)}{\cos z \cos x} = 0.$$

In Exercises 99–102, find the exact value of each expression. Do not use a calculator.

99. $\sin\left(\cos^{-1} \frac{1}{2} + \sin^{-1} \frac{3}{5}\right)$

100. $\sin\left[\sin^{-1} \frac{3}{5} - \cos^{-1}\left(-\frac{4}{5}\right)\right]$

101. $\cos\left(\tan^{-1} \frac{4}{3} + \cos^{-1} \frac{5}{13}\right)$

102. $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

In Exercises 103–105, write each trigonometric expression as an algebraic expression (that is, without any trigonometric functions). Assume that x and y are positive and in the domain of the given inverse trigonometric function.

103. $\cos(\sin^{-1} x - \cos^{-1} y)$

104. $\sin(\tan^{-1} x - \sin^{-1} y)$

105. $\tan(\sin^{-1} x + \cos^{-1} y)$

Group Exercise

106. Remembering the six sum and difference identities can be difficult. Did you have problems with some exercises because the identity you were using in your head turned out to be an incorrect formula? Are there easy ways to remember the six new identities presented in this section? Group members should address this question, considering one identity at a time. For each formula, list ways to make it easier to remember.

Preview Exercises

Exercises 107–109 will help you prepare for the material covered in the next section.

107. Give exact values for $\sin 30°$, $\cos 30°$, $\sin 60°$, and $\cos 60°$.

108. Use the appropriate values from Exercise 107 to answer each of the following.

a. Is $\sin 2 \cdot 30°$, or $\sin 60°$, equal to $2 \sin 30°$?

b. Is $\sin 2 \cdot 30°$, or $\sin 60°$, equal to $2 \sin 30° \cos 30°$?

109. Use appropriate values from Exercise 107 to answer each of the following.

a. Is $\cos (2 \cdot 30°)$, or $\cos 60°$, equal to $2 \cos 30°$?

b. Is $\cos (2 \cdot 30°)$, or $\cos 60°$, equal to $\cos^2 30° - \sin^2 30°$?

Section 5.3

Double-Angle, Power-Reducing, and Half-Angle Formulas

Objectives

1. Use the double-angle formulas.
2. Use the power-reducing formulas.
3. Use the half-angle formulas.

We have a long history of throwing things. Prior to 400 B.C., the Greeks competed in games that included discus throwing. In the seventeenth century, English soldiers organized cannonball-throwing competitions. In 1827, a Yale University student, disappointed over failing an exam, took out his frustrations at the passing of a collection plate in chapel. Seizing the monetary tray, he flung it in the direction of a large open space on campus. Yale students see this act of frustration as the origin of the Frisbee.

In this section, we develop other important classes of identities, called the double-angle, power-reducing, and half-angle formulas. We will see how one of these formulas can be used by athletes to increase throwing distance.

1 Use the double-angle formulas.

Double-Angle Formulas

A number of basic identities follow from the sum formulas for sine, cosine, and tangent. The first category of identities involves **double-angle formulas**.

Double-Angle Formulas

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

To prove each of these formulas, we replace α and β by θ in the sum formulas for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, and $\tan(\alpha + \beta)$.