Section 5.5

Trigonometric Equations

Objectives

1. Find all solutions of a trigonometric equation.
2. Solve equations with multiple angles.
3. Solve trigonometric equations quadratic in form.
4. Use factoring to separate different functions in trigonometric equations.
5. Use identities to solve trigonometric equations.
6. Use a calculator to solve trigonometric equations.

Exponential functions display the manic energies of uncontrolled growth. By contrast, trigonometric functions repeat their behavior. Do they embody in their regularity some basic rhythm of the universe? The cycles of periodic phenomena provide events that we can comfortably count on. When will the moon look just as it does at this moment? When can I count on 13.5 hours of daylight? When will my breathing be exactly as it is right now? Models with trigonometric functions embrace the periodic rhythms of our world. Equations containing trigonometric functions are used to answer questions about these models.

Trigonometric Equations and Their Solutions

1 Find all solutions of a trigonometric equation.

A **trigonometric equation** is an equation that contains a trigonometric expression with a variable, such as $\sin x$. We have seen that some trigonometric equations are identities, such as $\sin^2 x + \cos^2 x = 1$. These equations are true for every value of the variable for which the expressions are defined. In this section, we consider trigonometric equations that are true for only some values of the variable. The values that satisfy such an equation are its **solutions**. (There are trigonometric equations that have no solution.)

An example of a trigonometric equation is

$$\sin x = \tfrac{1}{2}.$$

A solution of this equation is $\frac{\pi}{6}$ because $\sin\frac{\pi}{6} = \frac{1}{2}$. By contrast, π is not a solution because $\sin \pi = 0 \neq \frac{1}{2}$.

Is $\frac{\pi}{6}$ the only solution of $\sin x = \frac{1}{2}$? The answer is no. Because of the periodic nature of the sine function, there are infinitely many values of x for which $\sin x = \frac{1}{2}$. **Figure 5.7** shows five of the solutions, including $\frac{\pi}{6}$, for $-\frac{3\pi}{2} \leq x \leq \frac{7\pi}{2}$. Notice that the x-coordinates of the points where the graph of $y = \sin x$ intersects the line $y = \frac{1}{2}$ are the solutions of the equation $\sin x = \frac{1}{2}$.

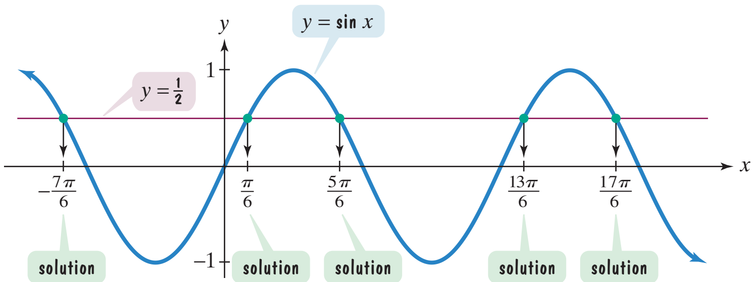


Figure 5.7 The equation $\sin x = \frac{1}{2}$ has five solutions when x is restricted to the interval $\left[-\frac{3\pi}{2}, \frac{7\pi}{2}\right]$.

How do we represent all solutions of $\sin x = \frac{1}{2}$? Because the period of the sine function is 2π, first find all solutions in $[0, 2\pi)$. The solutions are

$$x = \frac{\pi}{6} \quad \text{and} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

The sine is positive in quadrants I and II.

Any multiple of 2π can be added to these values and the sine is still $\frac{1}{2}$. Thus, all solutions of $\sin x = \frac{1}{2}$ are given by

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi,$$

where n is any integer. By choosing any two integers, such as $n = 0$ and $n = 1$, we can find some solutions of $\sin x = \frac{1}{2}$. Thus, four of the solutions are determined as follows:

Let $n = 0$.

$$x = \frac{\pi}{6} + 2 \cdot 0\pi = \frac{\pi}{6} \qquad x = \frac{5\pi}{6} + 2 \cdot 0\pi = \frac{5\pi}{6}$$

Let $n = 1$.

$$x = \frac{\pi}{6} + 2 \cdot 1\pi = \frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$$

$$x = \frac{5\pi}{6} + 2 \cdot 1\pi \qquad x = \frac{5\pi}{6} + 2\pi = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}.$$

These four solutions are shown among the five solutions in **Figure 5.7**.

Equations Involving a Single Trigonometric Function

To solve an equation containing a single trigonometric function:

- Isolate the function on one side of the equation.
- Solve for the variable.

EXAMPLE 1 Finding All Solutions of a Trigonometric Equation

Solve the equation: $3\sin x - 2 = 5\sin x - 1$.

Solution The equation contains a single trigonometric function, $\sin x$.

Step 1 Isolate the function on one side of the equation. We can solve for $\sin x$ by collecting terms with $\sin x$ on the left side and constant terms on the right side.

$$3\sin x - 2 = 5\sin x - 1 \qquad \text{This is the given equation.}$$

$$3\sin x - 5\sin x - 2 = 5\sin x - 5\sin x - 1 \qquad \text{Subtract 5 sin x from both sides.}$$

$$-2\sin x - 2 = -1 \qquad \text{Simplify.}$$

$$-2\sin x = 1 \qquad \text{Add 2 to both sides.}$$

$$\sin x = -\tfrac{1}{2} \qquad \text{Divide both sides by } -2 \text{ and solve for sin x.}$$

Step 2 Solve for the variable. We must solve for x in $\sin x = -\dfrac{1}{2}$. Because $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$, the solutions of $\sin x = -\dfrac{1}{2}$ in $[0, 2\pi)$ are

$$x = \pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6} \qquad x = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}.$$

The sine is negative in quadrant III.

The sine is negative in quadrant IV.

Because the period of the sine function is 2π, the solutions of the equation are given by

$$x = \frac{7\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{11\pi}{6} + 2n\pi,$$

where n is any integer.

Check Point 1 Solve the equation: $5 \sin x = 3 \sin x + \sqrt{3}$.

Now we will concentrate on finding solutions of trigonometric equations for $0 \leq x < 2\pi$. You can use a graphing utility to check the solutions of these equations. Graph the left side and graph the right side. The solutions are the x-coordinates of the points where the graphs intersect.

2 Solve equations with multiple angles.

Equations Involving Multiple Angles

Here are examples of two equations that include multiple angles:

$$\tan 3x = 1 \qquad \sin \frac{x}{2} = \frac{\sqrt{3}}{2}.$$

The angle is a multiple of 3.

The angle is a multiple of $\frac{1}{2}$.

We will solve each equation for $0 \leq x < 2\pi$. The period of the function plays an important role in ensuring that we do not leave out any solutions.

EXAMPLE 2 Solving an Equation with a Multiple Angle

Solve the equation: $\tan 3x = 1, \quad 0 \leq x < 2\pi$.

Solution The period of the tangent function is π. In the interval $[0, \pi)$, the only value for which the tangent function is 1 is $\frac{\pi}{4}$. This means that $3x = \frac{\pi}{4}$. Because the period is π, all the solutions to $\tan 3x = 1$ are given by

$$3x = \frac{\pi}{4} + n\pi. \qquad \text{n is any integer.}$$

$$x = \frac{\pi}{12} + \frac{n\pi}{3} \qquad \text{Divide both sides by 3 and solve for x.}$$

In the interval $[0, 2\pi)$, we obtain the solutions of $\tan 3x = 1$ as follows:

Let $n = 0$.
$$x = \frac{\pi}{12} + \frac{0\pi}{3} = \frac{\pi}{12}$$

Let $n = 1$.
$$x = \frac{\pi}{12} + \frac{1\pi}{3} = \frac{\pi}{12} + \frac{4\pi}{12} = \frac{5\pi}{12}$$

Let $n = 2$.
$$x = \frac{\pi}{12} + \frac{2\pi}{3} = \frac{\pi}{12} + \frac{8\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$$

Let $n = 3$.
$$x = \frac{\pi}{12} + \frac{3\pi}{3} = \frac{\pi}{12} + \frac{12\pi}{12} = \frac{13\pi}{12}$$

Let $n = 4$.
$$x = \frac{\pi}{12} + \frac{4\pi}{3} = \frac{\pi}{12} + \frac{16\pi}{12} = \frac{17\pi}{12}$$

Let $n = 5$.
$$x = \frac{\pi}{12} + \frac{5\pi}{3} = \frac{\pi}{12} + \frac{20\pi}{12} = \frac{21\pi}{12} = \frac{7\pi}{4}.$$

If you let $n = 6$, you will obtain $x = \frac{25\pi}{12}$. This value exceeds 2π. In the interval $[0, 2\pi)$, the solutions of $\tan 3x = 1$ are $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}$, and $\frac{7\pi}{4}$. These solutions are illustrated by the six intersection points in the technology box.

Technology

Graphic Connections

Shown below are the graphs of

$$y = \tan 3x$$

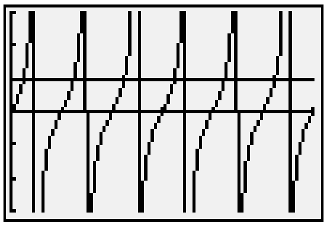
and

$$y = 1$$

in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-3, 3, 1]$ viewing rectangle. The solutions of

$$\tan 3x = 1$$

in $[0, 2\pi)$ are shown by the x-coordinates of the six intersection points.



Check Point 2 Solve the equation: $\tan 2x = \sqrt{3}, 0 \leq x < 2\pi$.

EXAMPLE 3 Solving an Equation with a Multiple Angle

Solve the equation: $\sin \frac{x}{2} = \frac{\sqrt{3}}{2}, 0 \leq x < 2\pi$.

Solution The period of the sine function is 2π. In the interval $[0, 2\pi)$, there are two values at which the sine function is $\frac{\sqrt{3}}{2}$. One of these values is $\frac{\pi}{3}$. The sine is positive in quadrant II; thus, the other value is $\pi - \frac{\pi}{3}$, or $\frac{2\pi}{3}$. This means that $\frac{x}{2} = \frac{\pi}{3}$ or $\frac{x}{2} = \frac{2\pi}{3}$. Because the period is 2π, all the solutions of $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$ are given by

$$\frac{x}{2} = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad \frac{x}{2} = \frac{2\pi}{3} + 2n\pi \qquad \text{n is any integer.}$$

$$x = \frac{2\pi}{3} + 4n\pi \qquad x = \frac{4\pi}{3} + 4n\pi. \qquad \text{Multiply both sides by 2 and solve for x.}$$

We see that $x = \frac{2\pi}{3} + 4n\pi$ or $x = \frac{4\pi}{3} + 4n\pi$. If $n = 0$, we obtain $x = \frac{2\pi}{3}$ from the first equation and $x = \frac{4\pi}{3}$ from the second equation. If we let $n = 1$, we are adding $4 \cdot 1 \cdot \pi$, or 4π, to $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. These values of x exceed 2π. Thus, in the interval $[0, 2\pi)$, the only solutions of $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Check Point 3 Solve the equation: $\sin\frac{x}{3} = \frac{1}{2}, 0 \le x < 2\pi$.

3 Solve trigonometric equations quadratic in form.

Trigonometric Equations Quadratic in Form

Some trigonometric equations are in the form of a quadratic equation $au^2 + bu + c = 0$, where u is a trigonometric function and $a \neq 0$. Here are two examples of trigonometric equations that are quadratic in form:

$$2\cos^2 x + \cos x - 1 = 0 \qquad 2\sin^2 x - 3\sin x + 1 = 0.$$

The form of this equation is $2u^2 + u - 1 = 0$ with $u = \cos x$.

The form of this equation is $2u^2 - 3u + 1 = 0$ with $u = \sin x$.

To solve this kind of equation, try using factoring. If the trigonometric expression does not factor, use another method, such as the quadratic formula or the square root property.

Technology

Graphic Connections

The graph of

$$y = 2\cos^2 x + \cos x - 1$$

is shown in a

$$\left[0, 2\pi, \frac{\pi}{2}\right] \text{ by } [-3, 3, 1]$$

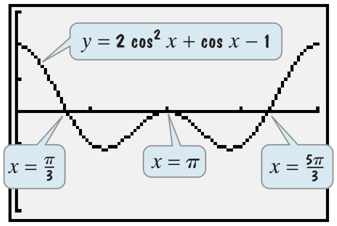
viewing rectangle. The x-intercepts,

$$\frac{\pi}{3}, \pi, \text{ and } \frac{5\pi}{3},$$

verify the three solutions of

$$2\cos^2 x + \cos x - 1 = 0$$

in $[0, 2\pi)$.



EXAMPLE 4 Solving a Trigonometric Equation Quadratic in Form

Solve the equation: $2\cos^2 x + \cos x - 1 = 0, \quad 0 \le x < 2\pi$.

Solution The given equation is in quadratic form $2u^2 + u - 1 = 0$ with $u = \cos x$. Let us attempt to solve the equation by factoring.

$$2\cos^2 x + \cos x - 1 = 0 \qquad \text{This is the given equation.}$$

$$(2\cos x - 1)(\cos x + 1) = 0 \qquad \text{Factor: Notice that } 2u^2 + u - 1 \text{ factors as } (2u - 1)(u + 1).$$

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0 \qquad \text{Set each factor equal to 0.}$$

$$2\cos x = 1 \qquad \cos x = -1 \qquad \text{Solve for cos x.}$$

$$\cos x = \tfrac{1}{2}$$

$$x = \frac{\pi}{3} \qquad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \qquad x = \pi \qquad \text{Solve each equation for x, } 0 \le x < 2\pi.$$

The cosine is positive in quadrants I and IV.

The solutions in the interval $[0, 2\pi,)$ are $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$.

Check Point 4 Solve the equation: $2\sin^2 x - 3\sin x + 1 = 0, \quad 0 \le x < 2\pi.$

EXAMPLE 5 Solving a Trigonometric Equation Quadratic in Form

Solve the equation: $4\sin^2 x - 1 = 0, \quad 0 \le x < 2\pi.$

Solution The given equation is in quadratic form $4u^2 - 1 = 0$ with $u = \sin x$. We can solve this equation by the square root property: If $u^2 = c$, then $u = \pm\sqrt{c}$.

$$4\sin^2 x - 1 = 0 \qquad \text{This is the given equation.}$$

$$4\sin^2 x = 1 \qquad \text{Add 1 to both sides.}$$

$$\sin^2 x = \frac{1}{4} \qquad \text{Divide both sides by 4 and solve for } \sin^2 x.$$

$$\sin x = \sqrt{\frac{1}{4}} = \frac{1}{2} \quad \text{or} \quad \sin x = -\sqrt{\frac{1}{4}} = -\frac{1}{2} \qquad \text{Apply the square root property: If } u^2 = c, \text{ then } u = \sqrt{c} \text{ or } u = -\sqrt{c}.$$

$$x = \frac{\pi}{6} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \quad x = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \quad x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \qquad \text{Solve each equation for } x, 0 \le x < 2\pi.$$

The sine is positive in quadrants I and II.

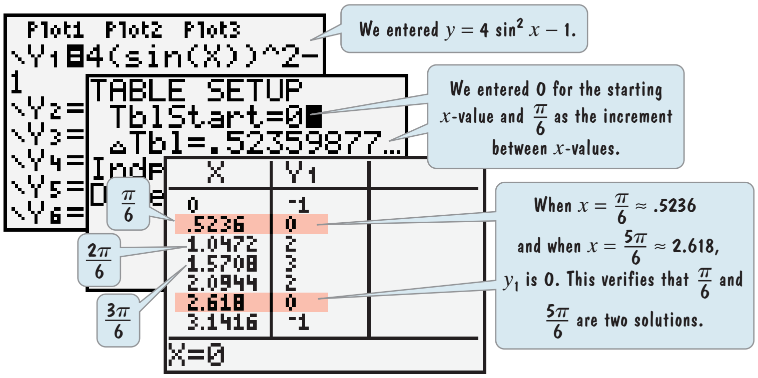
The sine is negative in quadrants III and IV.

The solutions in the interval $[0, 2\pi)$ are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

Technology

Numeric Connections

You can use a graphing utility's $\boxed{\text{TABLE}}$ feature to verify that the solutions of $4\sin^2 x - 1 = 0$ in $[0, 2\pi)$ are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. The table for $y = 4\sin^2 x - 1$, shown on the right, verifies that $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are solutions. Scroll through the table to verify the other two solutions.



Check Point 5 Solve the equation: $4\cos^2 x - 3 = 0, \quad 0 \le x < 2\pi.$

4 Use factoring to separate different functions in trigonometric equations.

Using Factoring to Separate Two Different Trigonometric Functions in an Equation

We have seen that factoring is used to solve some trigonometric equations that are quadratic in form. Factoring can also be used to solve some trigonometric equations that contain two different functions such as

$$\tan x \sin^2 x = 3\tan x.$$

In such a case, move all terms to one side and obtain zero on the other side. Then try to use factoring to separate the different functions. Example 6 shows how this is done.

Study Tip

In solving

$$\tan x \sin^2 x = 3 \tan x,$$

do not begin by dividing both sides by $\tan x$. Division by zero is undefined. If you divide by $\tan x$, you lose the two solutions for which $\tan x = 0$, namely 0 and π.

EXAMPLE 6 Using Factoring to Separate Different Functions

Solve the equation: $\tan x \sin^2 x = 3 \tan x, \quad 0 \le x < 2\pi$.

Solution Move all terms to one side and obtain zero on the other side.

$$\tan x \sin^2 x = 3 \tan x \quad \text{This is the given equation.}$$

$$\tan x \sin^2 x - 3 \tan x = 0 \quad \text{Subtract 3 tan x from both sides.}$$

We now have $\tan x \sin^2 x - 3 \tan x = 0$, which contains both tangent and sine functions. Use factoring to separate the two functions.

$$\tan x(\sin^2 x - 3) = 0 \quad \text{Factor out tan x from the two terms on the left side.}$$

$$\tan x = 0 \quad \text{or} \quad \sin^2 x - 3 = 0 \quad \text{Set each factor equal to 0.}$$

$$x = 0 \quad x = \pi \qquad \sin^2 x = 3 \quad \text{Solve for x.}$$

$$\sin x = \pm\sqrt{3}$$

This equation has no solution because $\sin x$ cannot be greater than 1 or less than −1.

The solutions in the interval $[0, 2\pi)$ are 0 and π.

Check Point 6 Solve the equation: $\sin x \tan x = \sin x, \quad 0 \le x < 2\pi$.

5 Use identities to solve trigonometric equations.

Using Identities to Solve Trigonometric Equations

Some trigonometric equations contain more than one function on the same side and these functions cannot be separated by factoring. For example, consider the equation

$$2 \cos^2 x + 3 \sin x = 0.$$

How can we obtain an equivalent equation that has only one trigonometric function? We use the identity $\sin^2 x + \cos^2 x = 1$ and substitute $1 - \sin^2 x$ for $\cos^2 x$. This forms the basis of our next example.

EXAMPLE 7 Using an Identity to Solve a Trigonometric Equation

Solve the equation: $2 \cos^2 x + 3 \sin x = 0, \quad 0 \le x < 2\pi$.

Solution

$$2 \cos^2 x + 3 \sin x = 0 \quad \text{This is the given equation.}$$

$$2(1 - \sin^2 x) + 3 \sin x = 0 \quad \cos^2 x = 1 - \sin^2 x$$

$$2 - 2 \sin^2 x + 3 \sin x = 0 \quad \text{Use the distributive property.}$$

$$-2 \sin^2 x + 3 \sin x + 2 = 0 \quad \text{Write the equation in descending powers of sin x.}$$

$$2 \sin^2 x - 3 \sin x - 2 = 0 \quad \text{Multiply both sides by } -1\text{. The equation is in quadratic form } 2u^2 - 3u - 2 = 0 \text{ with } u = \sin x.$$

It's easier to factor with a positive leading coefficient.

$$(2 \sin x + 1)(\sin x - 2) = 0 \quad \text{Factor. Notice that } 2u^2 - 3u - 2 \text{ factors as } (2u + 1)(u - 2).$$

$$2 \sin x + 1 = 0 \quad \text{or} \quad \sin x - 2 = 0 \quad \text{Set each factor equal to 0.}$$

$$2\sin x = -1 \qquad \sin x = 2$$ Solve $2\sin x + 1 = 0$ and $\sin x - 2 = 0$ for $\sin x$.

This equation has no solution because $\sin x$ cannot be greater than 1.

$$\sin x = -\frac{1}{2}$$

$$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \qquad x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$ Solve for x.

$\sin\frac{\pi}{6} = \frac{1}{2}$. The sine is negative in quadrants III and IV.

The solutions of $2\cos^2 x + 3\sin x = 0$ in the interval $[0, 2\pi)$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Check Point 7 Solve the equation: $2\sin^2 x - 3\cos x = 0, \quad 0 \le x < 2\pi$.

EXAMPLE 8 Using an Identity to Solve a Trigonometric Equation

Solve the equation: $\cos 2x + 3\sin x - 2 = 0, \quad 0 \le x < 2\pi$.

Solution The given equation contains a cosine function and a sine function. The cosine is a function of $2x$ and the sine is a function of x. We want one trigonometric function of the same angle. This can be accomplished by using the double-angle identity $\cos 2x = 1 - 2\sin^2 x$ to obtain an equivalent equation involving $\sin x$ only.

$\cos 2x + 3\sin x - 2 = 0$	This is the given equation.
$1 - 2\sin^2 x + 3\sin x - 2 = 0$	$\cos 2x = 1 - 2\sin^2 x$
$-2\sin^2 x + 3\sin x - 1 = 0$	Combine like terms.
$2\sin^2 x - 3\sin x + 1 = 0$	Multiply both sides by -1. The equation is in quadratic form $2u^2 - 3u + 1 = 0$ with $u = \sin x$.
$(2\sin x - 1)(\sin x - 1) = 0$	Factor. Notice that $2u^2 - 3u + 1$ factors as $(2u - 1)(u - 1)$.
$2\sin x - 1 = 0$ or $\sin x - 1 = 0$	Set each factor equal to 0.
$\sin x = \frac{1}{2}$ $\qquad$ $\sin x = 1$	Solve for $\sin x$.
$x = \frac{\pi}{6}$ $\quad$ $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ $\quad$ $x = \frac{\pi}{2}$	Solve each equation for x, $0 \le x < 2\pi$.

The sine is positive in quadrants I and II.

The solutions in the interval $[0, 2\pi)$ are $\frac{\pi}{6}, \frac{\pi}{2}$, and $\frac{5\pi}{6}$.

Check Point 8 Solve the equation: $\cos 2x + \sin x = 0, \quad 0 \le x < 2\pi$.

Sometimes it is necessary to do something to both sides of a trigonometric equation before using an identity. For example, consider the equation

$$\sin x \cos x = \tfrac{1}{2}.$$

This equation contains both a sine and a cosine function. How can we obtain a single function? Multiply both sides by 2. In this way, we can use the double-angle identity $\sin 2x = 2\sin x \cos x$ and obtain $\sin 2x$, a single function, on the left side.

Technology

Graphic Connections

Shown below are the graphs of

$$y = \sin x \cos x$$

and

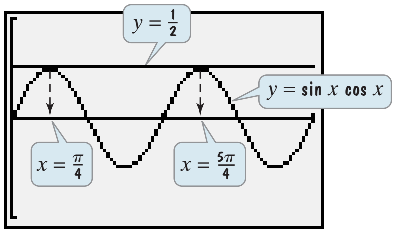
$$y = \tfrac{1}{2}$$

in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-1, 1, 1]$ viewing rectangle.

The solutions of

$$\sin x \cos x = \tfrac{1}{2}$$

are shown by the x-coordinates of the two intersection points.



EXAMPLE 9 Using an Identity to Solve a Trigonometric Equation

Solve the equation: $\sin x \cos x = \frac{1}{2}, \quad 0 \le x < 2\pi$.

Solution

$$\begin{aligned} \sin x \cos x &= \tfrac{1}{2} && \text{This is the given equation.} \\ 2 \sin x \cos x &= 1 && \text{Multiply both sides by 2 in anticipation of using } \sin 2x = 2 \sin x \cos x. \\ \sin 2x &= 1 && \text{Use a double-angle identity.} \end{aligned}$$

Notice that we have an equation, $\sin 2x = 1$, with $2x$, a multiple angle. The period of the sine function is 2π. In the interval $[0, 2\pi)$, the only value for which the sine function is 1 is $\frac{\pi}{2}$. This means that $2x = \frac{\pi}{2}$. Because the period is 2π, all the solutions of $\sin 2x = 1$ are given by

$$\begin{aligned} 2x &= \frac{\pi}{2} + 2n\pi && n \text{ is any integer.} \\ x &= \frac{\pi}{4} + n\pi && \text{Divide both sides by 2 and solve for } x. \end{aligned}$$

The solutions of $\sin x \cos x = \frac{1}{2}$ in the interval $[0, 2\pi)$ are obtained by letting $n = 0$ and $n = 1$. The solutions are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

Check Point 9 Solve the equation: $\sin x \cos x = -\frac{1}{2}, \quad 0 \le x < 2\pi$.

Let's look at another equation that contains two different functions, $\sin x - \cos x = 1$. Can you think of an identity that can be used to produce only one function? Perhaps $\sin^2 x + \cos^2 x = 1$ might be helpful. The next example shows how we can use this identity after squaring both sides of the given equation. Remember that if we raise both sides of an equation to an even power, we have the possibility of introducing extraneous solutions. Thus, we must check each proposed solution in the given equation. Alternatively, we can use a graphing utility to verify actual solutions.

Technology

Graphic Connections

A graphing utility can be used instead of an algebraic check. Shown are the graphs of

$$y = \sin x - \cos x$$

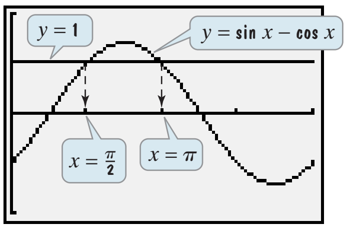
and

$$y = 1$$

in a $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $[-2, 2, 1]$ viewing rectangle. The actual solutions of

$$\sin x - \cos x = 1$$

are shown by the x-coordinates of the two intersection points, $\frac{\pi}{2}$ and π.



EXAMPLE 10 Using an Identity to Solve a Trigonometric Equation

Solve the equation: $\sin x - \cos x = 1, \quad 0 \le x < 2\pi$.

Solution We square both sides of the equation in anticipation of using $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned} \sin x - \cos x &= 1 && \text{This is the given equation.} \\ (\sin x - \cos x)^2 &= 1^2 && \text{Square both sides.} \\ \sin^2 x - 2 \sin x \cos x + \cos^2 x &= 1 && \text{Square the left side using } (A - B)^2 = A^2 - 2AB + B^2. \\ \sin^2 x + \cos^2 x - 2 \sin x \cos x &= 1 && \text{Rearrange terms.} \\ 1 - 2 \sin x \cos x &= 1 && \text{Apply a Pythagorean identity: } \sin^2 x + \cos^2 x = 1. \\ -2 \sin x \cos x &= 0 && \text{Subtract 1 from both sides of the equation.} \\ \sin x \cos x &= 0 && \text{Divide both sides of the equation by } -2. \\ \sin x = 0 \quad \text{or} \quad \cos x &= 0 && \text{Set each factor equal to 0.} \\ x = 0 \quad x = \pi \quad x = \frac{\pi}{2} \quad x &= \frac{3\pi}{2} && \text{Solve for } x \text{ in } [0, 2\pi). \end{aligned}$$

We check these proposed solutions to see if any are extraneous.

Check 0:

$\sin x - \cos x = 1$

$\sin 0 - \cos 0 \stackrel{?}{=} 1$

$0 - 1 \stackrel{?}{=} 1$

$-1 = 1$, false

0 is extraneous.

Check $\frac{\pi}{2}$:

$\sin x - \cos x = 1$

$\sin\frac{\pi}{2} - \cos\frac{\pi}{2} \stackrel{?}{=} 1$

$1 - 0 \stackrel{?}{=} 1$

$1 = 1$, true

Check π:

$\sin x - \cos x = 1$

$\sin \pi - \cos \pi \stackrel{?}{=} 1$

$0 - (-1) \stackrel{?}{=} 1$

$1 = 1$, true

Check $\frac{3\pi}{2}$:

$\sin x - \cos x = 1$

$\sin\frac{3\pi}{2} - \cos\frac{3\pi}{2} \stackrel{?}{=} 1$

$-1 - 0 \stackrel{?}{=} 1$

$-1 = 1$, false

$\frac{3\pi}{2}$ is extraneous.

The actual solutions of $\sin x - \cos x = 1$ in the interval $[0, 2\pi)$ are $\frac{\pi}{2}$ and π.

Check Point 10 Solve the equation: $\cos x - \sin x = -1, \quad 0 \le x < 2\pi$.

6 Use a calculator to solve trigonometric equations.

Using a Calculator to Solve Trigonometric Equations

In all our previous examples, the equations had solutions that were found by knowing the exact values of trigonometric functions of special angles, such as $\frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$. However, not all trigonometric equations involve these special angles. For those that do not, we will use the secondary keys marked $\boxed{\text{SIN}^{-1}}$, $\boxed{\text{COS}^{-1}}$, and $\boxed{\text{TAN}^{-1}}$ on a calculator. Recall that on most calculators, the inverse trigonometric function keys are the secondary functions for the buttons labeled $\boxed{\text{SIN}}$, $\boxed{\text{COS}}$, and $\boxed{\text{TAN}}$, respectively.

EXAMPLE 11 Solving Trigonometric Equations with a Calculator

Solve each equation, correct to four decimal places, for $0 \le x < 2\pi$:

a. $\tan x = 12.8044$ **b.** $\cos x = -0.4317$.

Solution We begin by using a calculator to find $\theta, 0 \le \theta < \frac{\pi}{2}$ satisfying the following equations:

$$\tan\theta = 12.8044 \qquad \cos\theta = 0.4317.$$

These numbers are the absolute values of the given range values.

Once θ is determined, we use our knowledge of the signs of the trigonometric functions to find x in $[0, 2\pi)$ satisfying $\tan x = 12.8044$ and $\cos x = -0.4317$.

Study Tip

To find solutions in $[0, 2\pi)$, your calculator must be in radian mode. Most scientific calculators revert to degree mode every time they are cleared.

a. $\tan x = 12.8044$ This is the given equation.

$\tan\theta = 12.8044$ Use a calculator to solve this equation for $\theta, 0 \le \theta < \frac{\pi}{2}$.

$\theta = \tan^{-1}(12.8044) \approx 1.4929$ 12.8044 $\boxed{\text{2nd}}$ $\boxed{\text{TAN}}$ or $\boxed{\text{2nd}}$ $\boxed{\text{TAN}}$ 12.8044 $\boxed{\text{ENTER}}$

$\tan x = 12.8044$ Return to the given equation. Because the tangent is positive, x lies in quadrant I or III.

$x \approx 1.4929 \qquad x \approx \pi + 1.4929 \approx 4.6345$ Solve for x, $0 \le x < 2\pi$.

The tangent is positive in quadrant I.

The tangent is positive in quadrant III.

Correct to four decimal places, the solutions of $\tan x = 12.8044$ in the interval $[0, 2\pi)$ are 1.4929 and 4.6345.

b. $\cos x = -0.4317$ — This is the given equation.

$\cos\theta = 0.4317$ — Use a calculator to solve this equation for $\theta, 0 \le \theta < \frac{\pi}{2}$.

$\theta = \cos^{-1}(0.4317) \approx 1.1244$ — .4317 $\boxed{\text{2nd}}$ $\boxed{\text{COS}}$ or $\boxed{\text{2nd}}$ $\boxed{\text{COS}}$ .4317 $\boxed{\text{ENTER}}$

$\cos x = -0.4317$ — Return to the given equation. Because the cosine is negative, x lies in quadrant II or III.

$x \approx \pi - 1.1244 \approx 2.0172$ $x \approx \pi + 1.1244 \approx 4.2660$ — Solve for x, $0 \le x < 2\pi$.

The cosine is negative in quadrant II.   The cosine is negative in quadrant III.

Correct to four decimal places, the solutions of $\cos x = -0.4317$ in the interval $[0, 2\pi)$ are 2.0172 and 4.2660.

Check Point 11 Solve each equation, correct to four decimal places, for $0 \le x < 2\pi$:

a. $\tan x = 3.1044$ **b.** $\sin x = -0.2315$.

EXAMPLE 12 Solving a Trigonometric Equation Using the Quadratic Formula and a Calculator

Solve the equation, correct to four decimal places, for $0 \le x < 2\pi$:

$$\sin^2 x - \sin x - 1 = 0.$$

Solution The given equation is in quadratic form $u^2 - u - 1 = 0$ with $u = \sin x$. We use the quadratic formula to solve for $\sin x$ because $u^2 - u - 1$ cannot be factored. Begin by identifying the values for a, b, and c.

$$\sin^2 x - \sin x - 1 = 0$$

$a = 1$ $b = -1$ $c = -1$

Substituting these values into the quadratic formula and simplifying gives the values for $\sin x$. Once we obtain these values, we will solve for x.

$$\sin x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1-(-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\sin x = \frac{1+\sqrt{5}}{2} \approx 1.6180 \quad \text{or} \quad \sin x = \frac{1-\sqrt{5}}{2} \approx -0.6180$$

This equation has no solution because $\sin x$ cannot be greater than 1.

The sine is negative in quadrants III and IV. Use a calculator to solve $\sin\theta = 0.6180$, $0 \le \theta < \frac{\pi}{2}$.

Using a calculator to solve $\sin\theta = 0.6180$, we have

$$\theta = \sin^{-1}(0.6180) \approx 0.6662.$$

We use 0.6662 to solve $\sin x = -0.6180$, $0 \le x < 2\pi$.

$$x \approx \pi + 0.6662 \approx 3.8078 \qquad x \approx 2\pi - 0.6662 \approx 5.6170$$

The sine is negative in quadrant III.   The sine is negative in quadrant IV.

Correct to four decimal places, the solutions of $\sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$ are 3.8078 and 5.6170.

Check Point 12 Solve the equation, correct to four decimal places, for $0 \le x < 2\pi$:

$$\cos^2 x + 5\cos x + 3 = 0.$$

Exercise Set 5.5

Practice Exercises

In Exercises 1–10, use substitution to determine whether the given x-value is a solution of the equation.

1. $\cos x = \frac{\sqrt{2}}{2}, \quad x = \frac{\pi}{4}$

2. $\tan x = \sqrt{3}, \quad x = \frac{\pi}{3}$

3. $\sin x = \frac{\sqrt{3}}{2}, \quad x = \frac{\pi}{6}$

4. $\sin x = \frac{\sqrt{2}}{2}, \quad x = \frac{\pi}{3}$

5. $\cos x = -\frac{1}{2}, \quad x = \frac{2\pi}{3}$

6. $\cos x = -\frac{1}{2}, \quad x = \frac{4\pi}{3}$

7. $\tan 2x = -\frac{\sqrt{3}}{3}, \quad x = \frac{5\pi}{12}$

8. $\cos\frac{2x}{3} = -\frac{1}{2}, \quad x = \pi$

9. $\cos x = \sin 2x, \quad x = \frac{\pi}{3}$

10. $\cos x + 2 = \sqrt{3}\sin x, \quad x = \frac{\pi}{6}$

In Exercises 11–24, find all solutions of each equation.

11. $\sin x = \frac{\sqrt{3}}{2}$

12. $\cos x = \frac{\sqrt{3}}{2}$

13. $\tan x = 1$

14. $\tan x = \sqrt{3}$

15. $\cos x = -\frac{1}{2}$

16. $\sin x = -\frac{\sqrt{2}}{2}$

17. $\tan x = 0$

18. $\sin x = 0$

19. $2\cos x + \sqrt{3} = 0$

20. $2\sin x + \sqrt{3} = 0$

21. $4\sin\theta - 1 = 2\sin\theta$

22. $5\sin\theta + 1 = 3\sin\theta$

23. $3\sin\theta + 5 = -2\sin\theta$

24. $7\cos\theta + 9 = -2\cos\theta$

Exercises 25–38 involve equations with multiple angles. Solve each equation on the interval $[0, 2\pi)$.

25. $\sin 2x = \frac{\sqrt{3}}{2}$

26. $\cos 2x = \frac{\sqrt{2}}{2}$

27. $\cos 4x = -\frac{\sqrt{3}}{2}$

28. $\sin 4x = -\frac{\sqrt{2}}{2}$

29. $\tan 3x = \frac{\sqrt{3}}{3}$

30. $\tan 3x = \sqrt{3}$

31. $\tan\frac{x}{2} = \sqrt{3}$

32. $\tan\frac{x}{2} = \frac{\sqrt{3}}{3}$

33. $\sin\frac{2\theta}{3} = -1$

34. $\cos\frac{2\theta}{3} = -1$

35. $\sec\frac{3\theta}{2} = -2$

36. $\cot\frac{3\theta}{2} = -\sqrt{3}$

37. $\sin\left(2x + \frac{\pi}{6}\right) = \frac{1}{2}$

38. $\sin\left(2x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Exercises 39–52 involve trigonometric equations quadratic in form. Solve each equation on the interval $[0, 2\pi)$.

39. $2\sin^2 x - \sin x - 1 = 0$

40. $2\sin^2 x + \sin x - 1 = 0$

41. $2\cos^2 x + 3\cos x + 1 = 0$

42. $\cos^2 x + 2\cos x - 3 = 0$

43. $2\sin^2 x = \sin x + 3$

44. $2\sin^2 x = 4\sin x + 6$

45. $\sin^2\theta - 1 = 0$

46. $\cos^2\theta - 1 = 0$

47. $4\cos^2 x - 1 = 0$

48. $4\sin^2 x - 3 = 0$

49. $9\tan^2 x - 3 = 0$

50. $3\tan^2 x - 9 = 0$

51. $\sec^2 x - 2 = 0$

52. $4\sec^2 x - 2 = 0$

In Exercises 53–62, solve each equation on the interval $[0, 2\pi)$.

53. $(\tan x - 1)(\cos x + 1) = 0$

54. $(\tan x + 1)(\sin x - 1) = 0$

55. $(2\cos x + \sqrt{3})(2\sin x + 1) = 0$

56. $(2\cos x - \sqrt{3})(2\sin x - 1) = 0$

57. $\cot x(\tan x - 1) = 0$

58. $\cot x(\tan x + 1) = 0$

59. $\sin x + 2\sin x\cos x = 0$

60. $\cos x - 2\sin x\cos x = 0$

61. $\tan^2 x\cos x = \tan^2 x$

62. $\cot^2 x\sin x = \cot^2 x$

In Exercises 63–84, use an identity to solve each equation on the interval $[0, 2\pi)$.

63. $2\cos^2 x + \sin x - 1 = 0$

64. $2\cos^2 x - \sin x - 1 = 0$

65. $\sin^2 x - 2\cos x - 2 = 0$

66. $4\sin^2 x + 4\cos x - 5 = 0$

67. $4\cos^2 x = 5 - 4\sin x$

68. $3\cos^2 x = \sin^2 x$

69. $\sin 2x = \cos x$

70. $\sin 2x = \sin x$

71. $\cos 2x = \cos x$

72. $\cos 2x = \sin x$

73. $\cos 2x + 5\cos x + 3 = 0$

74. $\cos 2x + \cos x + 1 = 0$

75. $\sin x\cos x = \frac{\sqrt{2}}{4}$

76. $\sin x\cos x = \frac{\sqrt{3}}{4}$

77. $\sin x + \cos x = 1$

78. $\sin x + \cos x = -1$

79. $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = 1$

80. $\sin\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{3}\right) = 1$

81. $\sin 2x\cos x + \cos 2x\sin x = \frac{\sqrt{2}}{2}$

82. $\sin 3x\cos 2x + \cos 3x\sin 2x = 1$

83. $\tan x + \sec x = 1$

84. $\tan x - \sec x = 1$

In Exercises 85–96, use a calculator to solve each equation, correct to four decimal places, on the interval $[0, 2\pi)$.

85. $\sin x = 0.8246$

86. $\sin x = 0.7392$

87. $\cos x = -\frac{2}{5}$

88. $\cos x = -\frac{4}{7}$

89. $\tan x = -3$

90. $\tan x = -5$

91. $\cos^2 x - \cos x - 1 = 0$

92. $3\cos^2 x - 8\cos x - 3 = 0$

93. $4\tan^2 x - 8\tan x + 3 = 0$

94. $\tan^2 x - 3\tan x + 1 = 0$

95. $7\sin^2 x - 1 = 0$

96. $5\sin^2 x - 1 = 0$

In Exercises 97–116, use the most appropriate method to solve each equation on the interval $[0, 2\pi)$. Use exact values where possible or give approximate solutions correct to four decimal places.

97. $2\cos 2x + 1 = 0$

98. $2\sin 3x + \sqrt{3} = 0$

99. $\sin 2x + \sin x = 0$

100. $\sin 2x + \cos x = 0$

101. $3\cos x - 6\sqrt{3} = \cos x - 5\sqrt{3}$

102. $\cos x - 5 = 3\cos x + 6$

103. $\tan x = -4.7143$

104. $\tan x = -6.2154$

105. $2\sin^2 x = 3 - \sin x$

106. $2\sin^2 x = 2 - 3\sin x$

107. $\cos x \csc x = 2\cos x$

108. $\tan x \sec x = 2\tan x$

109. $5\cot^2 x - 15 = 0$

110. $5\sec^2 x - 10 = 0$

111. $\cos^2 x + 2\cos x - 2 = 0$

112. $\cos^2 x + 5\cos x - 1 = 0$

113. $5\sin x = 2\cos^2 x - 4$

114. $7\cos x = 4 - 2\sin^2 x$

115. $2\tan^2 x + 5\tan x + 3 = 0$

116. $3\tan^2 x - \tan x - 2 = 0$

Practice Plus

In Exercises 117–120, graph f and g in the same rectangular coordinate system for $0 \le x \le 2\pi$. Then solve a trigonometric equation to determine points of intersection and identify these points on your graphs.

117. $f(x) = 3\cos x$, $g(x) = \cos x - 1$

118. $f(x) = 3\sin x$, $g(x) = \sin x - 1$

119. $f(x) = \cos 2x$, $g(x) = -2\sin x$

120. $f(x) = \cos 2x$, $g(x) = 1 - \sin x$

In Exercises 121–126, solve each equation on the interval $[0, 2\pi)$.

121. $|\cos x| = \frac{\sqrt{3}}{2}$

122. $|\sin x| = \frac{1}{2}$

123. $10\cos^2 x + 3\sin x - 9 = 0$

124. $3\cos^2 x - \sin x = \cos^2 x$

125. $2\cos^3 x + \cos^2 x - 2\cos x - 1 = 0$ (*Hint:* Use factoring by grouping.)

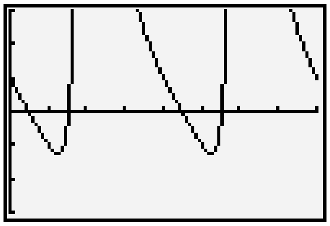
126. $2\sin^3 x - \sin^2 x - 2\sin x + 1 = 0$ (*Hint:* Use factoring by grouping.)

In Exercises 127–128, find the x-intercepts, correct to four decimal places, of the graph of each function. Then use the x-intercepts to match the function with its graph. The graphs are labeled (a) and (b).

127. $f(x) = \tan^2 x - 3\tan x + 1$

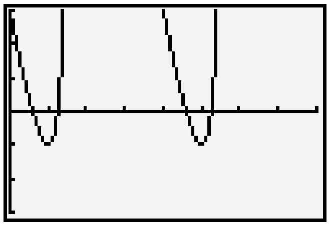
128. $g(x) = 4\tan^2 x - 8\tan x + 3$

a.



$[0, 2\pi, \frac{\pi}{4}]$ by $[-3, 3, 1]$

b.



$[0, 2\pi, \frac{\pi}{4}]$ by $[-3, 3, 1]$

Application Exercises

Use this information to solve Exercises 129–130. Our cycle of normal breathing takes place every 5 seconds. Velocity of air flow, y, measured in liters per second, after x seconds is modeled by

$$y = 0.6\sin\frac{2\pi}{5}x.$$

Velocity of air flow is positive when we inhale and negative when we exhale.

129. Within each breathing cycle, when are we inhaling at a rate of 0.3 liter per second? Round to the nearest tenth of a second.

130. Within each breathing cycle, when are we exhaling at a rate of 0.3 liter per second? Round to the nearest tenth of a second.

Use this information to solve Exercises 131–132. The number of hours of daylight in Boston is given by

$$y = 3\sin\left[\frac{2\pi}{365}(x - 79)\right] + 12,$$

where x is the number of days after January 1.

131. Within a year, when does Boston have 10.5 hours of daylight? Give your answer in days after January 1 and round to the nearest day.

132. Within a year, when does Boston have 13.5 hours of daylight? Give your answer in days after January 1 and round to the nearest day.

Use this information to solve Exercises 133–134. A ball on a spring is pulled 4 inches below its rest position and then released. After t seconds, the ball's distance, d, in inches from its rest position is given by

$$d = -4\cos\frac{\pi}{3}t.$$

133. Find all values of t for which the ball is 2 inches above its rest position.

134. Find all values of t for which the ball is 2 inches below its rest position.

Use this information to solve Exercises 135–136. When throwing an object, the distance achieved depends on its initial velocity, v_0, and the angle above the horizontal at which the object is thrown, θ. The distance, d, in feet, that describes the range covered is given by

$$d = \frac{v_0^2}{16}\sin\theta\cos\theta,$$

where v_0 is measured in feet per second.

135. You and your friend are throwing a baseball back and forth. If you throw the ball with an initial velocity of $v_0 = 90$ feet per second, at what angle of elevation, θ, to the nearest degree, should you direct your throw so that it can be easily caught by your friend located 170 feet away?

136. In Exercise 135, you increase the distance between you and your friend to 200 feet. With this increase, at what angle of elevation, θ, to the nearest degree, should you direct your throw?

Writing in Mathematics

137. What are the solutions of a trigonometric equation?

138. Describe the difference between verifying a trigonometric identity and solving a trigonometric equation.

139. Without actually solving the equation, describe how to solve

$$3\tan x - 2 = 5\tan x - 1.$$

140. In the interval $[0, 2\pi)$, the solutions of $\sin x = \cos 2x$ are $\frac{\pi}{6}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. Explain how to use graphs generated by a graphing utility to check these solutions.

141. Suppose you are solving equations in the interval $[0, 2\pi)$. Without actually solving equations, what is the difference between the number of solutions of $\sin x = \frac{1}{2}$ and $\sin 2x = \frac{1}{2}$? How do you account for this difference?

In Exercises 142–143, describe a general strategy for solving each equation. Do not solve the equation.

142. $2\sin^2 x + 5\sin x + 3 = 0$

143. $\sin 2x = \sin x$

144. Describe a natural periodic phenomenon. Give an example of a question that can be answered by a trigonometric equation in the study of this phenomenon.

145. A city's tall buildings and narrow streets reduce the amount of sunlight. If h is the average height of the buildings and w is the width of the street, the angle of elevation from the street to the top of the buildings is given by the trigonometric equation

$$\tan\theta = \frac{h}{w}.$$

A value of $\theta = 63°$ can result in an 85% loss of illumination. Some people experience depression with loss of sunlight. Determine whether such a person should live on a city street that is 80 feet wide with buildings whose heights average 400 feet. Explain your answer and include θ, to the nearest degree, in your argument.

Technology Exercises

146. Use a graphing utility to verify the solutions of any five equations that you solved in Exercises 63–84.

In Exercises 147–151, use a graphing utility to approximate the solutions of each equation in the interval $[0, 2\pi)$. Round to the nearest hundredth of a radian.

147. $15\cos^2 x + 7\cos x - 2 = 0$

148. $\cos x = x$

149. $2\sin^2 x = 1 - 2\sin x$

150. $\sin 2x = 2 - x^2$

151. $\sin x + \sin 2x + \sin 3x = 0$

Critical Thinking Exercises

Make Sense? *In Exercises 152–155, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

152. I solved $4\cos^2 x = 5 - 4\sin x$ by working independently with the left side, applying a Pythagorean identity, and transforming the left side into $5 - 4\sin x$.

153. There are similarities and differences between solving $4x + 1 = 3$ and $4\sin\theta + 1 = 3$: In the first equation, I need to isolate x to get the solution. In the trigonometric equation, I need to first isolate $\sin\theta$, but then I must continue to solve for θ.

154. I solved $\cos\left(x - \frac{\pi}{3}\right) = -1$ by first applying the formula for the cosine of the difference of two angles.

155. Using the equation for simple harmonic motion described in Exercises 133–134, I need to solve a trigonometric equation to determine the ball's distance from its rest position after 2 seconds.

In Exercises 156–159, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

156. The equation $(\sin x - 3)(\cos x + 2) = 0$ has no solution.

157. The equation $\tan x = \frac{\pi}{2}$ has no solution.

158. A trigonometric equation with an infinite number of solutions is an identity.

159. The equations $\sin 2x = 1$ and $\sin 2x = \frac{1}{2}$ have the same number of solutions on the interval $[0, 2\pi)$.

In Exercises 160–162, solve each equation on the interval $[0, 2\pi)$. Do not use a calculator.

160. $2\cos x - 1 + 3\sec x = 0$

161. $\sin 3x + \sin x + \cos x = 0$

162. $\sin x + 2\sin\frac{x}{2} = \cos\frac{x}{2} + 1$

Preview Exercises

Exercises 163–165 will help you prepare for the material covered in the first section of the next chapter. Solve each equation by using the cross-products principle to clear fractions from the proportion:

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc. \ (b \neq 0 \text{ and } d \neq 0)$$

Round to the nearest tenth.

163. Solve for a: $\dfrac{a}{\sin 46^\circ} = \dfrac{56}{\sin 63^\circ}$.

164. Solve for B, $0 < B < 180^\circ$: $\dfrac{81}{\sin 43^\circ} = \dfrac{62}{\sin B}$.

165. Solve for B: $\dfrac{51}{\sin 75^\circ} = \dfrac{71}{\sin B}$.

Chapter 5 Summary, Review, and Test

Summary

DEFINITIONS AND CONCEPTS	EXAMPLES
5.1 Verifying Trigonometric Identities	
a. Identities are trigonometric equations that are true for all values of the variable for which the expressions are defined.	
b. Fundamental trigonometric identities are given in the box on page 586.	
c. Guidelines for verifying trigonometric identities are given in the box on page 593.	Ex. 1, p. 587; Ex. 2, p. 588; Ex. 3, p. 588; Ex. 4, p. 589; Ex. 5, p. 590; Ex. 6, p. 590; Ex. 7, p. 591; Ex. 8, p. 592
5.2 Sum and Difference Formulas	
a. Sum and difference formulas are given in the box on page 599 and the box on page 603.	
b. Sum and difference formulas can be used to find exact values of trigonometric functions.	Ex. 1, p. 598; Ex. 2, p. 598; Ex. 4, p. 600; Ex. 5, p. 601
c. Sum and difference formulas can be used to verify trigonometric identities.	Ex. 3, p. 599; Ex. 6, p. 602; Ex. 7, p. 603
5.3 Double-Angle, Power-Reducing, and Half-Angle Formulas	
a. Double-angle, power-reducing, and half-angle formulas are given in the box on page 614.	
b. Double-angle and half-angle formulas can be used to find exact values of trigonometric functions.	Ex. 1, p. 608; Ex. 2, p. 609; Ex. 5, p. 612
c. Double-angle and half-angle formulas can be used to verify trigonometric identities.	Ex. 3, p. 609; Ex. 6, p. 613; Ex. 7, p. 613
d. Power-reducing formulas can be used to reduce the powers of trigonometric functions.	Ex. 4, p. 610
5.4 Product-to-Sum and Sum-to-Product Formulas	
a. The product-to-sum formulas are given in the box on page 619.	Ex. 1, p. 620
b. The sum-to-product formulas are given in the box on page 620. These formulas are useful to verify identities with fractions that contain sums and differences of sines and/or cosines.	Ex. 2, p. 621; Ex. 3, p. 621