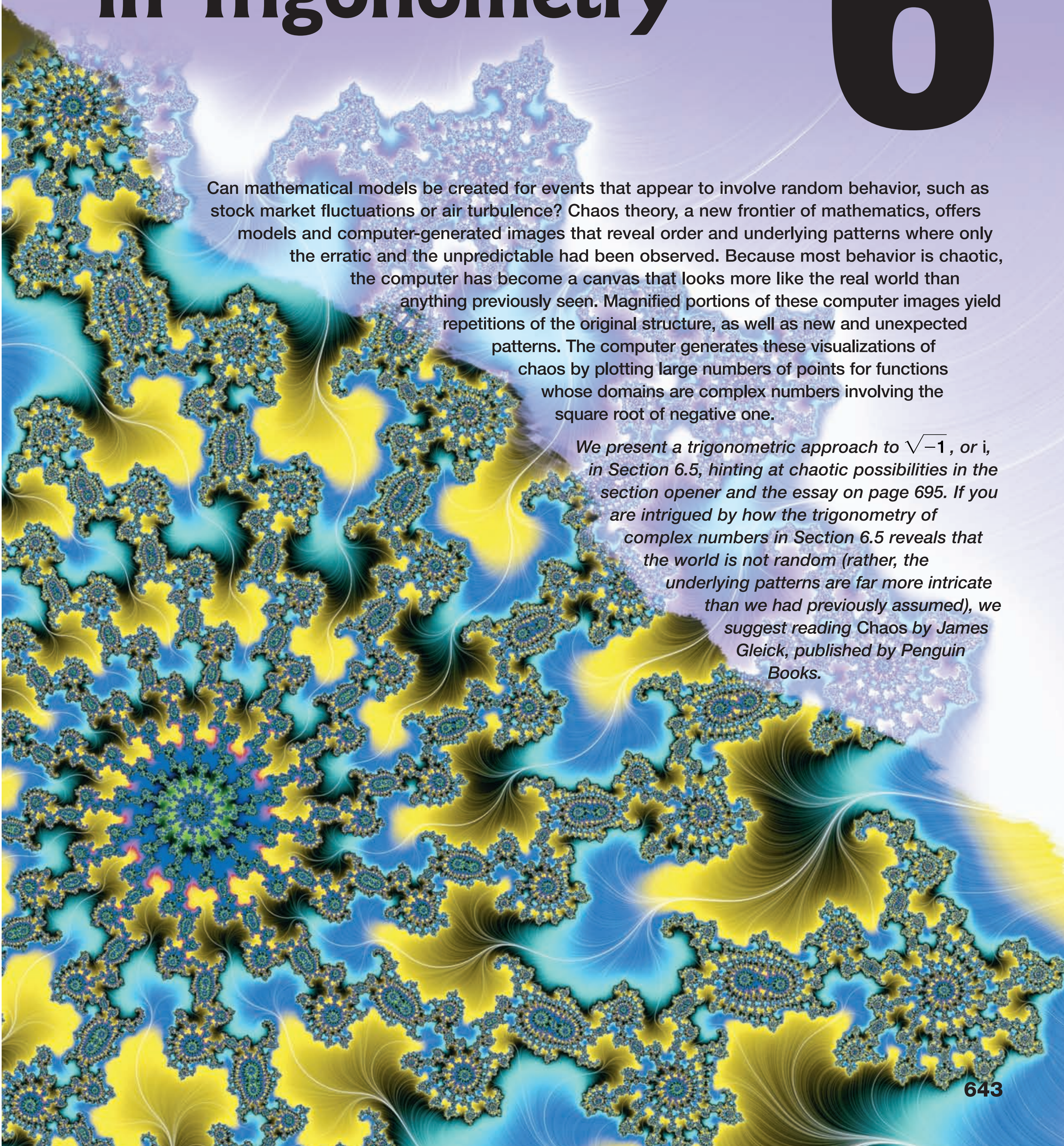
Additional Topics in Trigonometry

6



Can mathematical models be created for events that appear to involve random behavior, such as stock market fluctuations or air turbulence? Chaos theory, a new frontier of mathematics, offers models and computer-generated images that reveal order and underlying patterns where only the erratic and the unpredictable had been observed. Because most behavior is chaotic, the computer has become a canvas that looks more like the real world than anything previously seen. Magnified portions of these computer images yield repetitions of the original structure, as well as new and unexpected patterns. The computer generates these visualizations of chaos by plotting large numbers of points for functions whose domains are complex numbers involving the square root of negative one.

We present a trigonometric approach to $\sqrt{-1}$*, or* i*, in Section 6.5, hinting at chaotic possibilities in the section opener and the essay on page 695. If you are intrigued by how the trigonometry of complex numbers in Section 6.5 reveals that the world is not random (rather, the underlying patterns are far more intricate than we had previously assumed), we suggest reading* Chaos *by James Gleick, published by Penguin Books.*

Section 6.1

The Law of Sines

Objectives

1. Use the Law of Sines to solve oblique triangles.
2. Use the Law of Sines to solve, if possible, the triangle or triangles in the ambiguous case.
3. Find the area of an oblique triangle using the sine function.
4. Solve applied problems using the Law of Sines.

Point Reyes National Seashore, 40 miles north of San Francisco, consists of 75,000 acres with miles of pristine surf-pummeled beaches, forested ridges, and bays flanked by white cliffs. A few people, inspired by nature in the raw, live on private property adjoining the National Seashore. In 1995, a fire in the park burned 12,350 acres and destroyed 45 homes.

Fire is a necessary part of the life cycle in many wilderness areas. It is also an ongoing threat to those who choose to live surrounded by nature's unspoiled beauty. In this section, we see how trigonometry can be used to locate small wilderness fires before they become raging infernos. To do this, we begin by considering triangles other than right triangles.

The Law of Sines and Its Derivation

An **oblique triangle** is a triangle that does not contain a right angle. **Figure 6.1** shows that an oblique triangle has either three acute angles or two acute angles and one obtuse angle. Notice that the angles are labeled A, B, and C. The sides opposite each angle are labeled as a, b, and c, respectively.

Study Tip

Up until now, our work with triangles has involved right triangles. **Do not apply relationships that are valid for right triangles to oblique triangles.** Avoid the error of using the Pythagorean Theorem, $a^2 + b^2 = c^2$, to find a missing side of an oblique triangle. This relationship among the three sides applies only to right triangles.

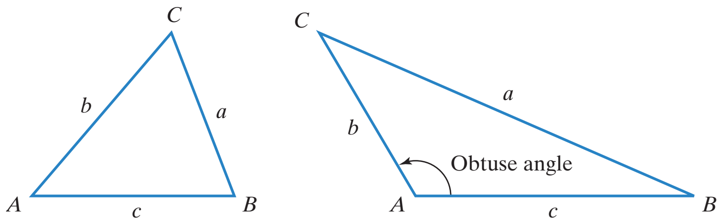


Figure 6.1 Oblique triangles

The relationships among the sides and angles of right triangles defined by the trigonometric functions are not valid for oblique triangles. Thus, we must observe and develop new relationships in order to work with oblique triangles.

Many relationships exist among the sides and angles in oblique triangles. One such relationship is called the **Law of Sines**.

The Law of Sines

If A, B, and C are the measures of the angles of a triangle, and a, b, and c are the lengths of the sides opposite these angles, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The ratio of the length of the side of any triangle to the sine of the angle opposite that side is the same for all three sides of the triangle.

Study Tip

The Law of Sines can be expressed with the sines in the numerator:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

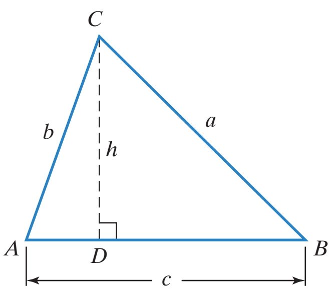


Figure 6.2 Drawing an altitude to prove the Law of Sines

To prove the Law of Sines, we draw an altitude of length h from one of the vertices of the triangle. In **Figure 6.2**, the altitude is drawn from vertex C. Two smaller triangles are formed, triangles ACD and BCD. Note that both are right triangles. Thus, we can use the definition of the sine of an angle of a right triangle.

$$\sin B = \frac{h}{a} \qquad \sin A = \frac{h}{b} \qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$h = a\sin B \qquad h = b\sin A \qquad \text{Solve each equation for } h.$$

Because we have found two expressions for h, we can set these expressions equal to each other.

$$a \sin B = b \sin A \quad \text{Equate the expressions for } h.$$

$$\frac{a \sin B}{\sin A \sin B} = \frac{b \sin A}{\sin A \sin B} \quad \text{Divide both sides by } \sin A \sin B.$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{Simplify.}$$

This proves part of the Law of Sines. If we use the same process and draw an altitude of length h from vertex A, we obtain the following result:

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

When this equation is combined with the previous equation, we obtain the Law of Sines. Because the sine of an angle is equal to the sine of 180° minus that angle, the Law of Sines is derived in a similar manner if the oblique triangle contains an obtuse angle.

1 Use the Law of Sines to solve oblique triangles.

Solving Oblique Triangles

Solving an oblique triangle means finding the lengths of its sides and the measurements of its angles. The Law of Sines can be used to solve a triangle in which one side and two angles are known. The three known measurements can be abbreviated using SAA (a side and two angles are known) or ASA (two angles and the side between them are known).

EXAMPLE 1 Solving an SAA Triangle Using the Law of Sines

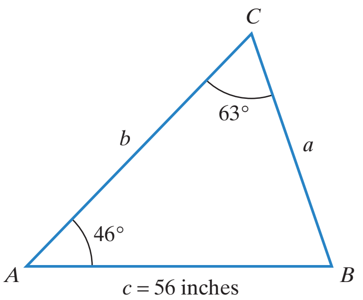


Figure 6.3 Solving an oblique SAA triangle

Solve the triangle shown in **Figure 6.3** with $A = 46°$, $C = 63°$, and $c = 56$ inches. Round lengths of sides to the nearest tenth.

Solution We begin by finding B, the third angle of the triangle. We do not need the Law of Sines to do this. Instead, we use the fact that the sum of the measures of the interior angles of a triangle is 180°.

$$A + B + C = 180°$$

$$46° + B + 63° = 180° \quad \text{Substitute the given values: } A = 46° \text{ and } C = 63°.$$

$$109° + B = 180° \quad \text{Add.}$$

$$B = 71° \quad \text{Subtract 109° from both sides.}$$

When we use the Law of Sines, we must be given one of the three ratios. In this example, we are given c and C: $c = 56$ and $C = 63°$. Thus, we use the ratio $\frac{c}{\sin C}$, or $\frac{56}{\sin 63°}$, to find the other two sides. Use the Law of Sines to find a.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{The ratio of any side to the sine of its opposite angle equals the ratio of any other side to the sine of its opposite angle.}$$

$$\frac{a}{\sin 46°} = \frac{56}{\sin 63°} \quad A = 46°,\ c = 56, \text{ and } C = 63°.$$

$$a = \frac{56 \sin 46°}{\sin 63°} \quad \text{Multiply both sides by } \sin 46° \text{ and solve for } a.$$

$$a \approx 45.2 \text{ inches} \quad \text{Use a calculator.}$$

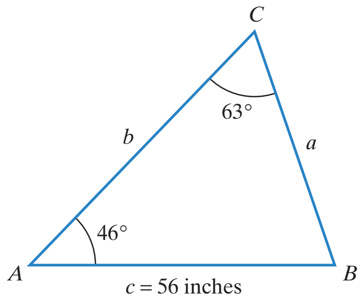


Figure 6.3 (repeated)

Use the Law of Sines again, this time to find b.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$ We use the given ratio, $\frac{c}{\sin C}$, to find b.

$$\frac{b}{\sin 71°} = \frac{56}{\sin 63°}$$ We found that $B = 71°$. We are given $c = 56$ and $C = 63°$.

$$b = \frac{56 \sin 71°}{\sin 63°}$$ Multiply both sides by sin 71° and solve for b.

$$b \approx 59.4 \text{ inches}$$ Use a calculator.

The solution is $B = 71°$, $a \approx 45.2$ inches, and $b \approx 59.4$ inches.

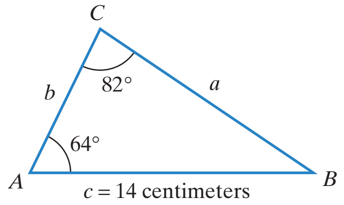


Figure 6.4

Check Point 1 Solve the triangle shown in **Figure 6.4** with $A = 64°$, $C = 82°$, and $c = 14$ centimeters. Round as in Example 1.

EXAMPLE 2 Solving an ASA Triangle Using the Law of Sines

Solve triangle ABC if $A = 50°$, $C = 33.5°$, and $b = 76$. Round measures to the nearest tenth.

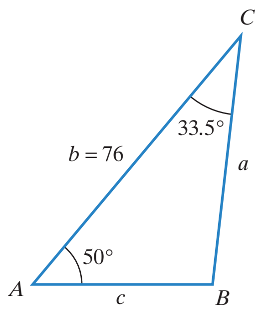


Figure 6.5 Solving an ASA triangle

Solution We begin by drawing a picture of triangle ABC and labeling it with the given information. **Figure 6.5** shows the triangle that we must solve. We begin by finding B.

$$A + B + C = 180°$$ The sum of the measures of a triangle's interior angles is 180°.

$$50° + B + 33.5° = 180°$$ $A = 50°$ and $C = 33.5°$.

$$83.5° + B = 180°$$ Add.

$$B = 96.5°$$ Subtract 83.5° from both sides.

Keep in mind that we must be given one of the three ratios to apply the Law of Sines. In this example, we are given that $b = 76$ and we found that $B = 96.5°$. Thus, we use the ratio $\frac{b}{\sin B}$, or $\frac{76}{\sin 96.5°}$, to find the other two sides. Use the Law of Sines to find a and c.

Find a: | **Find c:**

This is the known ratio.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 50°} = \frac{76}{\sin 96.5°} \qquad \frac{c}{\sin 33.5°} = \frac{76}{\sin 96.5°}$$

$$a = \frac{76 \sin 50°}{\sin 96.5°} \approx 58.6 \qquad c = \frac{76 \sin 33.5°}{\sin 96.5°} \approx 42.2$$

The solution is $B = 96.5°$, $a \approx 58.6$, and $c \approx 42.2$.

Check Point 2 Solve triangle ABC if $A = 40°$, $C = 22.5°$, and $b = 12$. Round as in Example 2.

2 Use the Law of Sines to solve, if possible, the triangle or triangles in the ambiguous case.

The Ambiguous Case (SSA)

If we are given two sides and an angle opposite one of them (SSA), does this determine a unique triangle? Can we solve this case using the Law of Sines? Such a case is called the **ambiguous case** because the given information may result in one triangle, two triangles, or no triangle at all. For example, in **Figure 6.6**, we are given a, b, and A. Because a is shorter than h, it is not long enough to form a triangle. The number of possible triangles, if any, that can be formed in the SSA case depends on h, the length of the altitude, where $h = b \sin A$.

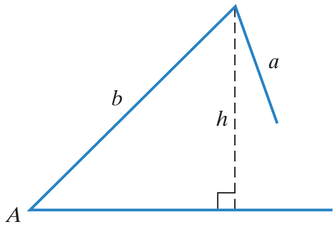
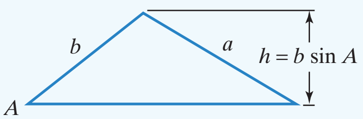


Figure 6.6 Given SSA, no triangle may result.

The Ambiguous Case (SSA)

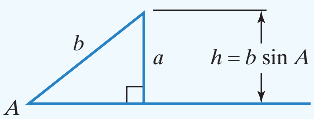
Consider a triangle in which a, b, and A are given. This information may result in

One Triangle



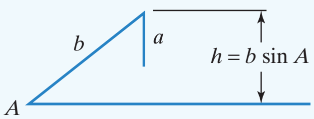
a is greater than h and a is greater than b. One triangle is formed.

One Right Triangle



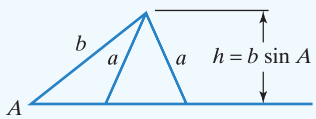
$a = h$ and is just the right length to form a right triangle.

No Triangle



a is less than h and is not long enough to form a triangle.

Two Triangles



a is greater than h and a is less than b. Two distinct triangles are formed.

In an SSA situation, it is not necessary to draw an accurate sketch like those shown in the box. The Law of Sines determines the number of triangles, if any, and gives the solution for each triangle.

EXAMPLE 3 Solving an SSA Triangle Using the Law of Sines (One Solution)

Solve triangle ABC if $A = 43°$, $a = 81$, and $b = 62$. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

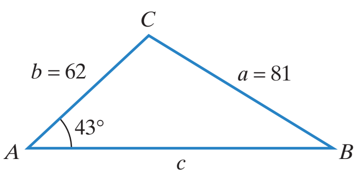


Figure 6.7 Solving an SSA triangle; the ambiguous case

Solution We begin with the sketch in **Figure 6.7**. The known ratio is $\frac{a}{\sin A}$, or $\frac{81}{\sin 43°}$. Because side b is given, we use the Law of Sines to find angle B.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{Apply the Law of Sines.}$$

$$\frac{81}{\sin 43°} = \frac{62}{\sin B} \quad a = 81,\ b = 62, \text{ and } A = 43°.$$

$$81 \sin B = 62 \sin 43° \quad \text{Cross multiply: If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

$$\sin B = \frac{62 \sin 43°}{81} \quad \text{Divide both sides by 81 and solve for sin } B.$$

$$\sin B \approx 0.5220 \quad \text{Use a calculator.}$$

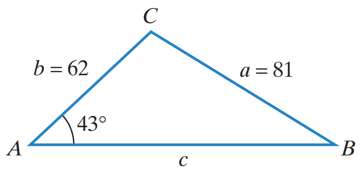


Figure 6.7 (repeated)

There are two angles B between 0° and 180° for which $\sin B \approx 0.5220$.

$$B_1 \approx 31° \qquad B_2 \approx 180° - 31° = 149°$$

Obtain the acute angle with your calculator in degree mode: $\sin^{-1} 0.5220$.

The sine is positive in quadrant II.

Look at **Figure 6.7**. Given that $A = 43°$, can you see that $B_2 \approx 149°$ is impossible? By adding 149° to the given angle, 43°, we exceed a 180° sum:

$$43° + 149° = 192°.$$

Thus, the only possibility is that $B_1 \approx 31°$. We find C using this approximation for B_1 and the measure that was given for A: $A = 43°$.

$$C = 180° - B_1 - A \approx 180° - 31° - 43° = 106°$$

Side c that lies opposite this 106° angle can now be found using the Law of Sines.

$$\frac{c}{\sin C} = \frac{a}{\sin A} \qquad \text{Apply the Law of Sines.}$$

$$\frac{c}{\sin 106°} = \frac{81}{\sin 43°} \qquad a = 81,\ C \approx 106°, \text{ and } A = 43°.$$

$$c = \frac{81 \sin 106°}{\sin 43°} \approx 114.2 \qquad \text{Multiply both sides by } \sin 106° \text{ and solve for } c.$$

There is one triangle and the solution is B_1(or B) $\approx 31°$, $C \approx 106°$, and $c \approx 114.2$.

Check Point 3 Solve triangle ABC if $A = 57°$, $a = 33$, and $b = 26$. Round as in Example 3.

EXAMPLE 4 Solving an SSA Triangle Using the Law of Sines (No Solution)

Solve triangle ABC if $A = 75°$, $a = 51$, and $b = 71$.

Solution The known ratio is $\frac{a}{\sin A}$, or $\frac{51}{\sin 75°}$. Because side b is given, we use the Law of Sines to find angle B.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \text{Use the Law of Sines.}$$

$$\frac{51}{\sin 75°} = \frac{71}{\sin B} \qquad \text{Substitute the given values.}$$

$$51 \sin B = 71 \sin 75° \qquad \text{Cross multiply: If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

$$\sin B = \frac{71 \sin 75°}{51} \approx 1.34 \qquad \text{Divide by 51 and solve for } \sin B.$$

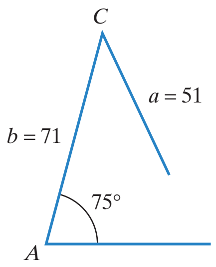


Figure 6.8 a is not long enough to form a triangle.

Because the sine can never exceed 1, there is no angle B for which $\sin B \approx 1.34$. There is no triangle with the given measurements, as illustrated in **Figure 6.8**.

Check Point 4 Solve triangle ABC if $A = 50°$, $a = 10$, and $b = 20$.

EXAMPLE 5 Solving an SSA Triangle Using the Law of Sines (Two Solutions)

Solve triangle ABC if $A = 40°$, $a = 54$, and $b = 62$. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

Solution The known ratio is $\frac{a}{\sin A}$, or $\frac{54}{\sin 40°}$. We use the Law of Sines to find angle B.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$ Use the Law of Sines.

$$\frac{54}{\sin 40°} = \frac{62}{\sin B}$$ Substitute the given values.

$$54 \sin B = 62 \sin 40°$$ Cross multiply: If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

$$\sin B = \frac{62 \sin 40°}{54} \approx 0.7380$$ Divide by 54 and solve for sin B.

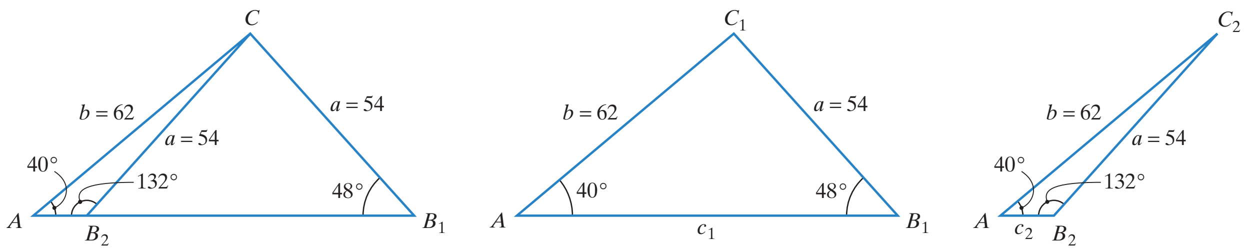
There are two angles B between 0° and 180° for which $\sin B \approx 0.7380$.

$$B_1 \approx 48° \qquad B_2 \approx 180° - 48° = 132°$$

Find $\sin^{-1} 0.7380$ with your calculator.

The sine is positive in quadrant II.

If you add either angle to the given angle, 40°, the sum does not exceed 180°. Thus, there are two triangles with the given conditions, shown in **Figure 6.9(a)**. The triangles, AB_1C_1 and AB_2C_2, are shown separately in **Figure 6.9(b)** and **Figure 6.9(c)**.



(a) Two triangles are possible with $A = 40°$, $a = 54$, and $b = 62$.

(b) In one possible triangle, $B_1 = 48°$.

(c) In the second possible triangle, $B_2 = 132°$.

Figure 6.9

Study Tip

The two triangles shown in **Figure 6.9** are helpful in organizing the solutions. However, if you keep track of the two triangles, one with the given information and $B_1 = 48°$, and the other with the given information and $B_2 = 132°$, you do not have to draw the figure to solve the triangles.

We find angles C_1 and C_2 using a 180° angle sum in each of the two triangles.

$$C_1 = 180° - A - B_1 \approx 180° - 40° - 48° = 92°$$

$$C_2 = 180° - A - B_2 \approx 180° - 40° - 132° = 8°$$

We use the Law of Sines to find c_1 and c_2.

$$\frac{c_1}{\sin C_1} = \frac{a}{\sin A} \qquad \frac{c_2}{\sin C_2} = \frac{a}{\sin A}$$

$$\frac{c_1}{\sin 92°} = \frac{54}{\sin 40°} \qquad \frac{c_2}{\sin 8°} = \frac{54}{\sin 40°}$$

$$c_1 = \frac{54 \sin 92°}{\sin 40°} \approx 84.0 \qquad c_2 = \frac{54 \sin 8°}{\sin 40°} \approx 11.7$$

There are two triangles. In one triangle, the solution is $B_1 \approx 48°$, $C_1 \approx 92°$, and $c_1 \approx 84.0$. In the other triangle, $B_2 \approx 132°$, $C_2 \approx 8°$, and $c_2 \approx 11.7$.

Check Point 5 Solve triangle ABC if $A = 35°$, $a = 12$, and $b = 16$. Round as in Example 5.

3 Find the area of an oblique triangle using the sine function.

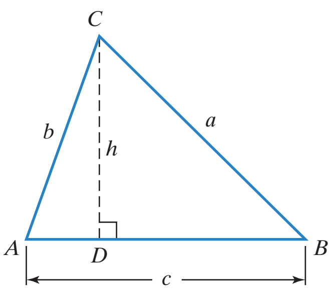


Figure 6.10

The Area of an Oblique Triangle

A formula for the area of an oblique triangle can be obtained using the procedure for proving the Law of Sines. We draw an altitude of length h from one of the vertices of the triangle, as shown in **Figure 6.10**. We apply the definition of the sine of angle A, $\frac{\text{opposite}}{\text{hypotenuse}}$, in right triangle ACD:

$$\sin A = \frac{h}{b}, \quad \text{so} \quad h = b \sin A.$$

The area of a triangle is $\frac{1}{2}$ the product of any side and the altitude drawn to that side. Using the altitude h in **Figure 6.10**, we have

$$\text{Area} = \frac{1}{2}ch = \frac{1}{2}cb \sin A.$$

Use the result from above: $h = b \sin A$.

This result, Area $= \frac{1}{2}cb \sin A$, or $\frac{1}{2}bc \sin A$, indicates that the area of the triangle is one-half the product of b and c times the sine of their included angle. If we draw altitudes from the other two vertices, we see that we can use any two sides to compute the area.

Area of an Oblique Triangle

The area of a triangle equals one-half the product of the lengths of two sides times the sine of their included angle. In **Figure 6.10**, this wording can be expressed by the formulas

$$\text{Area} = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ab \sin C = \tfrac{1}{2}ac \sin B.$$

EXAMPLE 6 Finding the Area of an Oblique Triangle

Find the area of a triangle having two sides of lengths 24 meters and 10 meters and an included angle of 62°. Round to the nearest square meter.

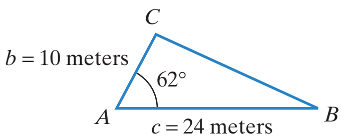


Figure 6.11 Finding the area of an SAS triangle

Solution The triangle is shown in **Figure 6.11**. Its area is half the product of the lengths of the two sides times the sine of the included angle.

$$\text{Area} = \tfrac{1}{2}(24)(10)(\sin 62°) \approx 106$$

The area of the triangle is approximately 106 square meters.

Check Point 6 Find the area of a triangle having two sides of lengths 8 meters and 12 meters and an included angle of 135°. Round to the nearest square meter.

4 Solve applied problems using the Law of Sines.

Applications of the Law of Sines

We have seen how the trigonometry of right triangles can be used to solve many different kinds of applied problems. The Law of Sines enables us to work with triangles that are not right triangles. As a result, this law can be used to solve problems involving surveying, engineering, astronomy, navigation, and the environment. Example 7 illustrates the use of the Law of Sines in detecting potentially devastating fires.

EXAMPLE 7 An Application of the Law of Sines

Two fire-lookout stations are 20 miles apart, with station B directly east of station A. Both stations spot a fire on a mountain to the north. The bearing from station A to the fire is N50°E (50° east of north). The bearing from station B to the fire is N36°W (36° west of north). How far, to the nearest tenth of a mile, is the fire from station A?

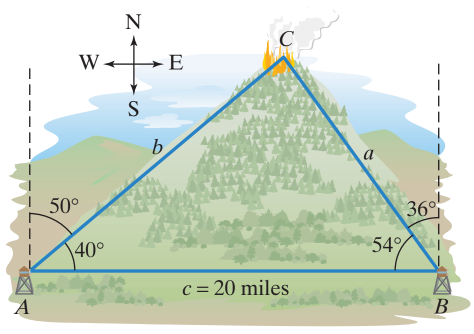


Figure 6.12

Solution **Figure 6.12** shows the information given in the problem. The distance from station A to the fire is represented by b. Notice that the angles describing the bearing from each station to the fire, 50° and 36°, are not interior angles of triangle ABC. Using a north-south line, the interior angles are found as follows:

$$A = 90° - 50° = 40° \qquad B = 90° - 36° = 54°.$$

To find b using the Law of Sines, we need a known side and an angle opposite that side. Because $c = 20$ miles, we find angle C using a 180° angle sum in the triangle. Thus,

$$C = 180° - A - B = 180° - 40° - 54° = 86°.$$

The ratio $\frac{c}{\sin C}$, or $\frac{20}{\sin 86°}$, is now known. We use this ratio and the Law of Sines to find b.

$$\frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{Use the Law of Sines.}$$

$$\frac{b}{\sin 54°} = \frac{20}{\sin 86°} \qquad c = 20,\ B = 54°,\text{ and } C = 86°.$$

$$b = \frac{20 \sin 54°}{\sin 86°} \approx 16.2 \qquad \text{Multiply both sides by } \sin 54° \text{ and solve for } b.$$

The fire is approximately 16.2 miles from station A.

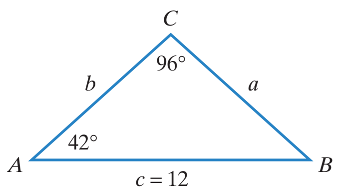
Check Point 7 Two fire-lookout stations are 13 miles apart, with station B directly east of station A. Both stations spot a fire. The bearing of the fire from station A is N35°E and the bearing of the fire from station B is N49°W. How far, to the nearest tenth of a mile, is the fire from station B?

Exercise Set 6.1

Practice Exercises

In Exercises 1–8, solve each triangle. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

1.



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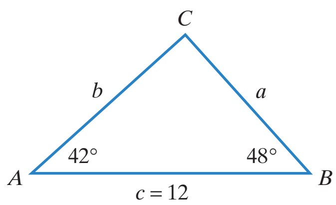

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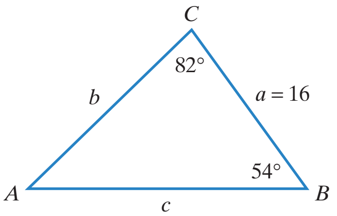

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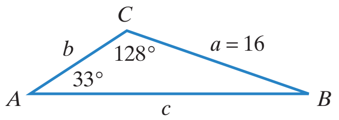

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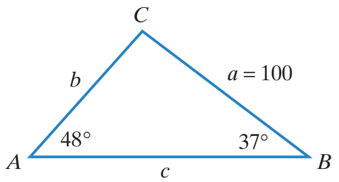

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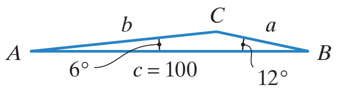

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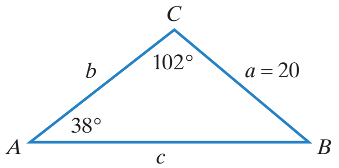

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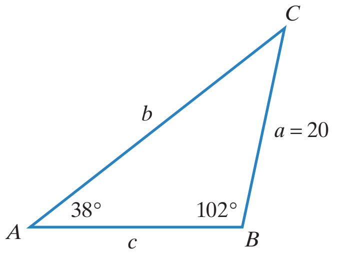


In Exercises 9–16, solve each triangle. Round lengths to the nearest tenth and angle measures to the nearest degree.

9. $A = 44°, B = 25°, a = 12$

10. $A = 56°, C = 24°, a = 22$

11. $B = 85°, C = 15°, b = 40$

12. $A = 85°, B = 35°, c = 30$

13. $A = 115°, C = 35°, c = 200$

14. $B = 5°, C = 125°, b = 200$

15. $A = 65°, B = 65°, c = 6$

16. $B = 80°, C = 10°, a = 8$

In Exercises 17–32, two sides and an angle (SSA) of a triangle are given. Determine whether the given measurements produce one triangle, two triangles, or no triangle at all. Solve each triangle that results. Round to the nearest tenth and the nearest degree for sides and angles, respectively.

17. $a = 20, b = 15, A = 40°$

18. $a = 30, b = 20, A = 50°$

19. $a = 10, c = 8.9, A = 63°$

20. $a = 57.5, c = 49.8, A = 136°$

21. $a = 42.1, c = 37, A = 112°$

22. $a = 6.1, b = 4, A = 162°$

23. $a = 10, b = 40, A = 30°$

24. $a = 10, b = 30, A = 150°$

25. $a = 16, b = 18, A = 60°$

26. $a = 30, b = 40, A = 20°$

27. $a = 12, b = 16.1, A = 37°$

28. $a = 7, b = 28, A = 12°$

29. $a = 22, c = 24.1, A = 58°$

30. $a = 95, c = 125, A = 49°$

31. $a = 9.3, b = 41, A = 18°$

32. $a = 1.4, b = 2.9, A = 142°$

In Exercises 33–38, find the area of the triangle having the given measurements. Round to the nearest square unit.

33. $A = 48°, b = 20$ feet, $c = 40$ feet

34. $A = 22°, b = 20$ feet, $c = 50$ feet

35. $B = 36°, a = 3$ yards, $c = 6$ yards

36. $B = 125°, a = 8$ yards, $c = 5$ yards

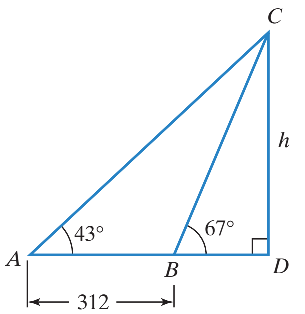
37. $C = 124°, a = 4$ meters, $b = 6$ meters

38. $C = 102°, a = 16$ meters, $b = 20$ meters

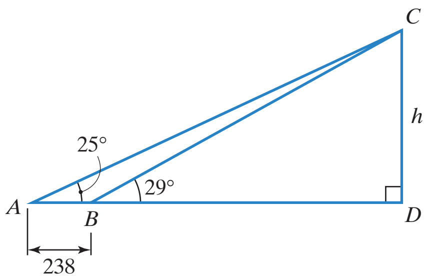
Practice Plus

In Exercises 39–40, find h to the nearest tenth.

39.

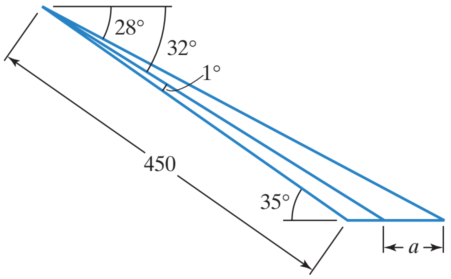


40.

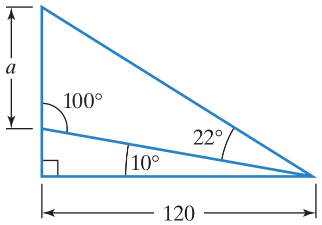


In Exercises 41–42, find a to the nearest tenth.

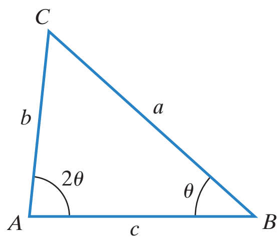
41.



42.



In Exercises 43–44, use the given measurements to solve the following triangle. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.



43. $a = 300, b = 200$

44. $a = 400, b = 300$

In Exercises 45–46, find the area of the triangle with the given vertices. Round to the nearest square unit.

45. $(-3, -2), (2, -2), (1, 2)$

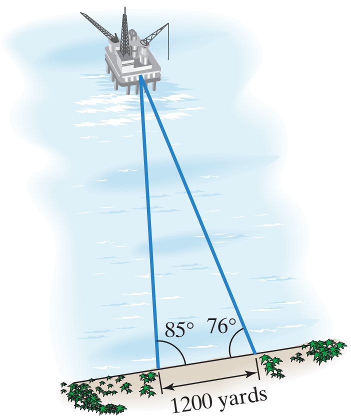
46. $(-2, -3), (-2, 2), (2, 1)$

Application Exercises

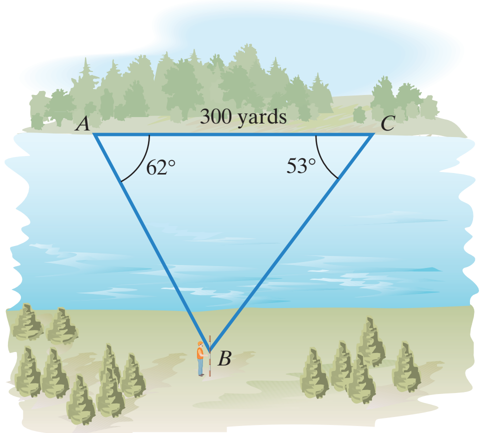
47. Two fire-lookout stations are 10 miles apart, with station B directly east of station A. Both stations spot a fire. The bearing of the fire from station A is N25°E and the bearing of the fire from station B is N56°W. How far, to the nearest tenth of a mile, is the fire from each lookout station?

48. The Federal Communications Commission is attempting to locate an illegal radio station. It sets up two monitoring stations, A and B, with station B 40 miles east of station A. Station A measures the illegal signal from the radio station as coming from a direction of 48° east of north. Station B measures the signal as coming from a point 34° west of north. How far is the illegal radio station from monitoring stations A and B? Round to the nearest tenth of a mile.

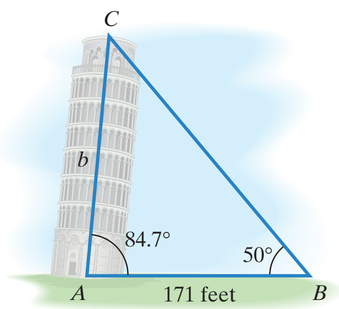
49. The figure shows a 1200-yard-long sand beach and an oil platform in the ocean. The angle made with the platform from one end of the beach is 85° and from the other end is 76°. Find the distance of the oil platform, to the nearest tenth of a yard, from each end of the beach.



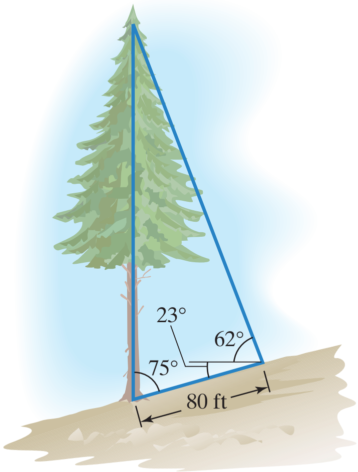
50. A surveyor needs to determine the distance between two points that lie on opposite banks of a river. The figure shows that 300 yards are measured along one bank. The angles from each end of this line segment to a point on the opposite bank are 62° and 53°. Find the distance between *A* and *B* to the nearest tenth of a yard.



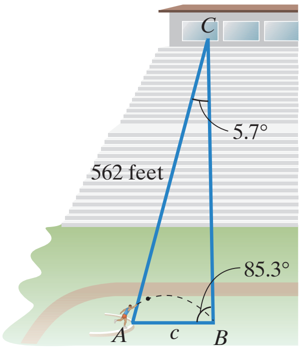
51. The Leaning Tower of Pisa in Italy leans at an angle of about 84.7°. The figure shows that 171 feet from the base of the tower, the angle of elevation to the top is 50°. Find the distance, to the nearest tenth of a foot, from the base to the top of the tower.



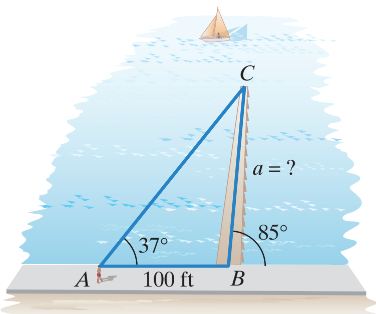
52. A pine tree growing on a hillside makes a 75° angle with the hill. From a point 80 feet up the hill, the angle of elevation to the top of the tree is 62° and the angle of depression to the bottom is 23°. Find, to the nearest tenth of a foot, the height of the tree.



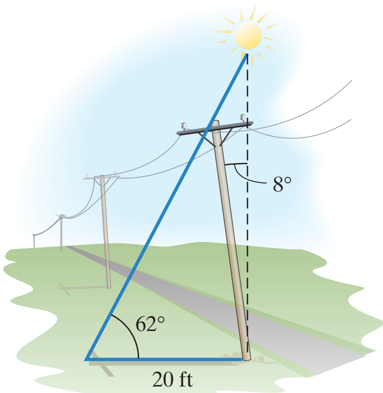
53. The figure shows a shot-put ring. The shot is tossed from A and lands at B. Using modern electronic equipment, the distance of the toss can be measured without the use of measuring tapes. When the shot lands at B, an electronic transmitter placed at B sends a signal to a device in the official's booth above the track. The device determines the angles at B and C. At a track meet, the distance from the official's booth to the shot-put ring is 562 feet. If $B = 85.3°$ and $C = 5.7°$, determine the length of the toss to the nearest tenth of a foot.



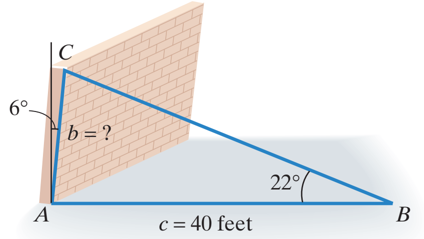
54. A pier forms an 85° angle with a straight shore. At a distance of 100 feet from the pier, the line of sight to the tip forms a 37° angle. Find the length of the pier to the nearest tenth of a foot.



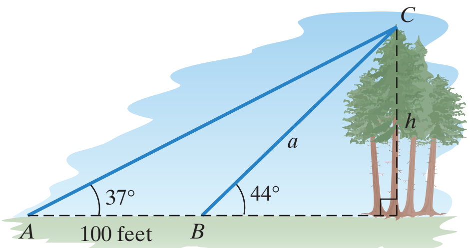
55. When the angle of elevation of the sun is 62°, a telephone pole that is tilted at an angle of 8° directly away from the sun casts a shadow 20 feet long. Determine the length of the pole to the nearest tenth of a foot.



56. A leaning wall is inclined 6° from the vertical. At a distance of 40 feet from the wall, the angle of elevation to the top is 22°. Find the height of the wall to the nearest tenth of a foot.



57. Redwood trees in California's Redwood National Park are hundreds of feet tall. The height of one of these trees is represented by h in the figure shown.



a. Use the measurements shown to find a, to the nearest tenth of a foot, in oblique triangle ABC.

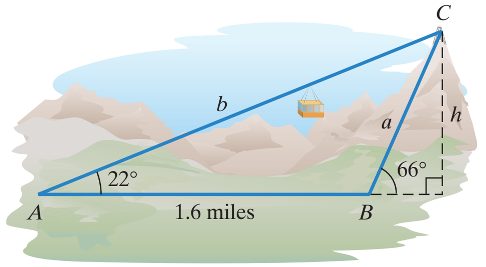
b. Use the right triangle shown to find the height, to the nearest tenth of a foot, of a typical redwood tree in the park.

58. The figure at the top of the next page shows a cable car that carries passengers from A to C. Point A is 1.6 miles from the base of the mountain. The angles of elevation from A and B to the mountain's peak are 22° and 66°, respectively.

a. Determine, to the nearest tenth of a foot, the distance covered by the cable car.

b. Find a, to the nearest tenth of a foot, in oblique triangle ABC.

c. Use the right triangle to find the height of the mountain to the nearest tenth of a foot.



59. Lighthouse B is 7 miles west of lighthouse A. A boat leaves A and sails 5 miles. At this time, it is sighted from B. If the bearing of the boat from B is N62°E, how far from B is the boat? Round to the nearest tenth of a mile.

60. After a wind storm, you notice that your 16-foot flagpole may be leaning, but you are not sure. From a point on the ground 15 feet from the base of the flagpole, you find that the angle of elevation to the top is 48°. Is the flagpole leaning? If so, find the acute angle, to the nearest degree, that the flagpole makes with the ground.

Writing in Mathematics

61. What is an oblique triangle?

62. Without using symbols, state the Law of Sines in your own words.

63. Briefly describe how the Law of Sines is proved.

64. What does it mean to solve an oblique triangle?

65. What do the abbreviations SAA and ASA mean?

66. Why is SSA called the ambiguous case?

67. How is the sine function used to find the area of an oblique triangle?

68. Write an original problem that can be solved using the Law of Sines. Then solve the problem.

69. Use Exercise 53 to describe how the Law of Sines is used for throwing events at track and field meets. Why aren't tape measures used to determine tossing distance?

70. You are cruising in your boat parallel to the coast, looking at a lighthouse. Explain how you can use your boat's speed and a device for measuring angles to determine the distance at any instant from your boat to the lighthouse.

Critical Thinking Exercises

Make Sense? *In Exercises 71–74, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

71. I began using the Law of Sines to solve an oblique triangle in which the measures of two sides and the angle between them were known.

72. If I know the measures of the sides and angles of an oblique triangle, I have three ways of determining the triangle's area.

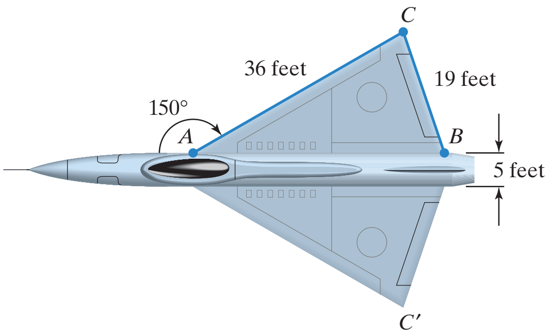
73. When solving an SSA triangle using the Law of Sines, my calculator gave me both the acute and obtuse angles B for which $\sin B = 0.5833$.

74. Under certain conditions, a fire can be located by superimposing a triangle onto the situation and applying the Law of Sines.

75. If you are given two sides of a triangle and their included angle, you can find the triangle's area. Can the Law of Sines be used to solve the triangle with this given information? Explain your answer.

76. Two buildings of equal height are 800 feet apart. An observer on the street between the buildings measures the angles of elevation to the tops of the buildings as 27° and 41°, respectively. How high, to the nearest foot, are the buildings?

77. The figure shows the design for the top of the wing of a jet fighter. The fuselage is 5 feet wide. Find the wing span CC' to the nearest tenth of a foot.



Preview Exercises

Exercises 78–80 will help you prepare for the material covered in the next section.

78. Find the obtuse angle B, rounded to the nearest degree, satisfying

$$\cos B = \frac{6^2 + 4^2 - 9^2}{2 \cdot 6 \cdot 4}.$$

79. Simplify and round to the nearest whole number:

$$\sqrt{26(26 - 12)(26 - 16)(26 - 24)}.$$

80. Two airplanes leave an airport at the same time on different runways. The first plane, flying on a bearing of N66°W, travels 650 miles after two hours. The second plane, flying on a bearing of S26°W, travels 600 miles after two hours. Illustrate the situation with an oblique triangle that shows how far apart the airplanes will be after two hours.