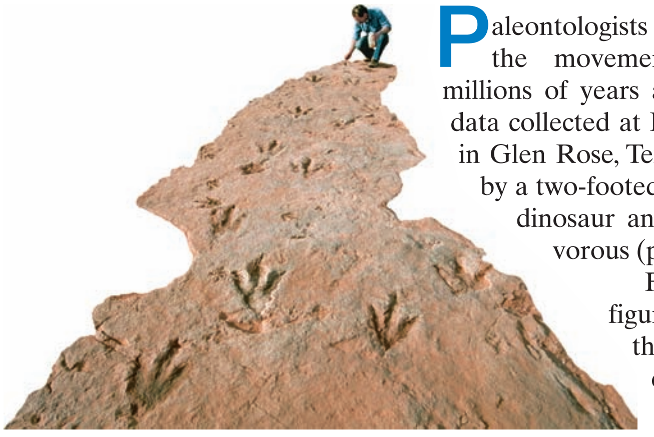
Section 6.2

The Law of Cosines

Objectives

1. Use the Law of Cosines to solve oblique triangles.
2. Solve applied problems using the Law of Cosines.
3. Use Heron's formula to find the area of a triangle.



Paleontologists use trigonometry to study the movements made by dinosaurs millions of years ago. **Figure 6.13**, based on data collected at Dinosaur Valley State Park in Glen Rose, Texas, shows footprints made by a two-footed carnivorous (meat-eating) dinosaur and the hindfeet of a herbivorous (plant-eating) dinosaur.

For each dinosaur, the figure indicates the *pace* and the *stride*. The pace is the distance from the left footprint to the right footprint, and vice versa. The stride is the distance from the left footprint to the next left footprint or from the right footprint to the next right footprint. Also shown in **Figure 6.13** is the pace angle, designated by θ. Notice that neither dinosaur moves with a pace angle of 180°, meaning that the footprints are directly in line. The footprints show a "zig-zig" pattern that is numerically described by the pace angle. A dinosaur that is an efficient walker has a pace angle close to 180°, minimizing zig-zag motion and maximizing forward motion.

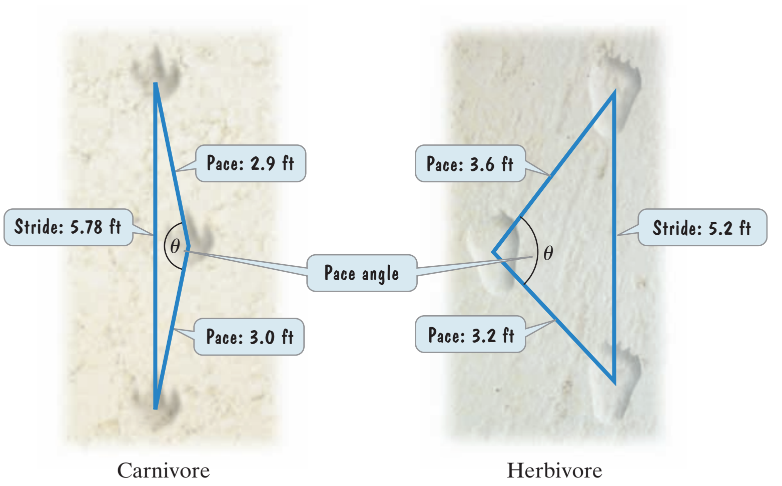


Figure 6.13 Dinosaur Footprints
Source: Glen J. Kuban, *An Overview of Dinosaur Tracking*

How can we determine the pace angles for the carnivore and the herbivore in **Figure 6.13**? Problems such as this, in which we know the measures of three sides of a triangle and we need to find the measurement of a missing angle, cannot be solved by the Law of Sines. To numerically describe which dinosaur in **Figure 6.13** made more forward progress with each step, we turn to the Law of Cosines.

The Law of Cosines and Its Derivation

We now look at another relationship that exists among the sides and angles in an oblique triangle. **The Law of Cosines** is used to solve triangles in which two sides and the included angle (SAS) are known, or those in which three sides (SSS) are known.

Discovery

What happens to the Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

if $C = 90°$? What familiar theorem do you obtain?

The Law of Cosines

If A, B, and C are the measures of the angles of a triangle, and a, b, and c are the lengths of the sides opposite these angles, then

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

The square of a side of a triangle equals the sum of the squares of the other two sides minus twice their product times the cosine of their included angle.

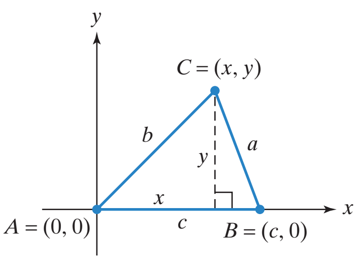


Figure 6.14

To prove the Law of Cosines, we place triangle ABC in a rectangular coordinate system. **Figure 6.14** shows a triangle with three acute angles. The vertex A is at the origin and side c lies along the positive x-axis. The coordinates of C are (x, y). Using the right triangle that contains angle A, we apply the definitions of the cosine and the sine.

$$\cos A = \frac{x}{b} \qquad \sin A = \frac{y}{b}$$

$$x = b\cos A \qquad y = b\sin A$$ Multiply both sides of each equation by b and solve for x and y, respectively.

Thus, the coordinates of C are $(x, y) = (b\cos A, b\sin A)$. Although triangle ABC in **Figure 6.14** shows angle A as an acute angle, if A were obtuse, the coordinates of C would still be $(b\cos A, b\sin A)$. This means that our proof applies to both kinds of oblique triangles.

We now apply the distance formula to the side of the triangle with length a. Notice that a is the distance from (x, y) to $(c, 0)$.

$$a = \sqrt{(x-c)^2 + (y-0)^2}$$ Use the distance formula.

$$a^2 = (x-c)^2 + y^2$$ Square both sides of the equation.

$$a^2 = (b\cos A - c)^2 + (b\sin A)^2$$ $x = b\cos A$ and $y = b\sin A$.

$$a^2 = b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$$ Square the two expressions.

$$a^2 = b^2\sin^2 A + b^2\cos^2 A + c^2 - 2bc\cos A$$ Rearrange terms.

$$a^2 = b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc\cos A$$ Factor b^2 from the first two terms.

$$a^2 = b^2 + c^2 - 2bc\cos A$$ $\sin^2 A + \cos^2 A = 1$

The resulting equation is one of the three formulas for the Law of Cosines. The other two formulas are derived in a similar manner.

1 Use the Law of Cosines to solve oblique triangles.

Solving Oblique Triangles

If you are given two sides and an included angle (SAS) of an oblique triangle, none of the three ratios in the Law of Sines is known. This means that we do not begin solving the triangle using the Law of Sines. Instead, we apply the Law of Cosines and the following procedure:

Solving an SAS Triangle

1. Use the Law of Cosines to find the side opposite the given angle.
2. Use the Law of Sines to find the angle opposite the shorter of the two given sides. This angle is always acute.
3. Find the third angle by subtracting the measure of the given angle and the angle found in step 2 from 180°.

EXAMPLE 1 Solving an SAS Triangle

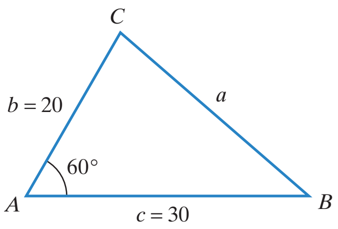


Figure 6.15 Solving an SAS triangle

Solve the triangle in **Figure 6.15** with $A = 60°$, $b = 20$, and $c = 30$. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

Solution We are given two sides and an included angle. Therefore, we apply the three-step procedure for solving an SAS triangle.

Step 1 Use the Law of Cosines to find the side opposite the given angle. Thus, we will find a.

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{Apply the Law of Cosines to find } a.$$

$$a^2 = 20^2 + 30^2 - 2(20)(30)\cos 60° \quad b = 20,\ c = 30, \text{ and } A = 60°.$$

$$= 400 + 900 - 1200(0.5) \quad \text{Perform the indicated operations.}$$

$$= 700$$

$$a = \sqrt{700} \approx 26.5 \quad \text{Take the square root of both sides and solve for } a.$$

Step 2 Use the Law of Sines to find the angle opposite the shorter of the two given sides. This angle is always acute. The shorter of the two given sides is $b = 20$. Thus, we will find acute angle B.

$$\frac{b}{\sin B} = \frac{a}{\sin A} \quad \text{Apply the Law of Sines.}$$

$$\frac{20}{\sin B} = \frac{\sqrt{700}}{\sin 60°} \quad \text{We are given } b = 20 \text{ and } A = 60°. \text{ Use the exact value of } a, \sqrt{700}, \text{ from step 1.}$$

$$\sqrt{700}\sin B = 20\sin 60° \quad \text{Cross multiply: If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

$$\sin B = \frac{20\sin 60°}{\sqrt{700}} \approx 0.6547 \quad \text{Divide by } \sqrt{700} \text{ and solve for } \sin B.$$

$$B \approx 41° \quad \text{Find } \sin^{-1} 0.6547 \text{ using a calculator.}$$

Step 3 Find the third angle. Subtract the measure of the given angle and the angle found in step 2 from 180°.

$$C = 180° - A - B \approx 180° - 60° - 41° = 79°$$

The solution is $a \approx 26.5$, $B \approx 41°$, and $C \approx 79°$.

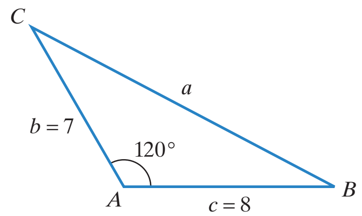


Figure 6.16

Check Point 1 Solve the triangle shown in **Figure 6.16** with $A = 120°$, $b = 7$, and $c = 8$. Round as in Example 1.

If you are given three sides of a triangle (SSS), solving the triangle involves finding the three angles. We use the following procedure:

Solving an SSS Triangle

1. Use the Law of Cosines to find the angle opposite the longest side.
2. Use the Law of Sines to find either of the two remaining acute angles.
3. Find the third angle by subtracting the measures of the angles found in steps 1 and 2 from 180°.

EXAMPLE 2 Solving an SSS Triangle

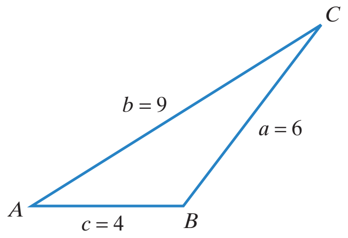


Figure 6.17 Solving an SSS triangle

Solve triangle ABC if $a = 6, b = 9,$ and $c = 4$. Round angle measures to the nearest degree.

Solution We are given three sides. Therefore, we apply the three-step procedure for solving an SSS triangle. The triangle is shown in **Figure 6.17**.

Step 1 Use the Law of Cosines to find the angle opposite the longest side. The longest side is $b = 9$. Thus, we will find angle B.

$$b^2 = a^2 + c^2 - 2ac \cos B$$ Apply the Law of Cosines to find B.

$$2ac \cos B = a^2 + c^2 - b^2$$ Solve for cos B.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{6^2 + 4^2 - 9^2}{2 \cdot 6 \cdot 4} = -\frac{29}{48}$$ $a = 6$, $b = 9$, and $c = 4$.

Using a calculator, $\cos^{-1}\left(\frac{29}{48}\right) \approx 53°$. Because $\cos B$ is negative, B is an obtuse angle. Thus,

$$B \approx 180° - 53° = 127°.$$

Because the domain of $y = \cos^{-1}x$ is $[0, \pi]$, you can use a calculator to find $\cos^{-1}\left(-\frac{29}{48}\right) \approx 127°$.

Study Tip

You can use the Law of Cosines in step 2 to find either of the remaining angles. However, it is simpler to use the Law of Sines. Because the largest angle has been found, the remaining angles must be acute. Thus, there is no need to be concerned about two possible triangles or an ambiguous case.

Step 2 Use the Law of Sines to find either of the two remaining acute angles. We will find angle A.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$ Apply the Law of Sines.

$$\frac{6}{\sin A} = \frac{9}{\sin 127°}$$ We are given $a = 6$ and $b = 9$. We found that $B \approx 127°$.

$$9 \sin A = 6 \sin 127°$$ Cross multiply.

$$\sin A = \frac{6 \sin 127°}{9} \approx 0.5324$$ Divide by 9 and solve for sin A.

$$A \approx 32°$$ Find $\sin^{-1} 0.5324$ using a calculator.

Step 3 Find the third angle. Subtract the measures of the angles found in steps 1 and 2 from 180°.

$$C = 180° - B - A \approx 180° - 127° - 32° = 21°$$

The solution is $B \approx 127°$, $A \approx 32°$, and $C \approx 21°$.

Check Point 2 Solve triangle ABC if $a = 8, b = 10,$ and $c = 5$. Round angle measures to the nearest degree.

2 Solve applied problems using the Law of Cosines.

Applications of the Law of Cosines

Applied problems involving SAS and SSS triangles can be solved using the Law of Cosines.

EXAMPLE 3 An Application of the Law of Cosines

Two airplanes leave an airport at the same time on different runways. One flies on a bearing of N66°W at 325 miles per hour. The other airplane flies on a bearing of S26°W at 300 miles per hour. How far apart will the airplanes be after two hours?

Solution After two hours, the plane flying at 325 miles per hour travels $325 \cdot 2$ miles, or 650 miles. Similarly, the plane flying at 300 miles per hour travels 600 miles. The situation is illustrated in **Figure 6.18**.

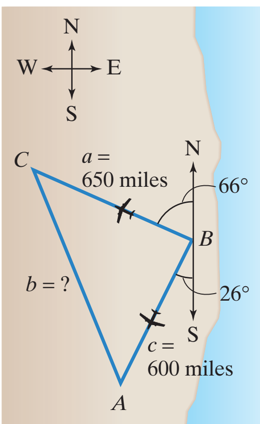


Figure 6.18

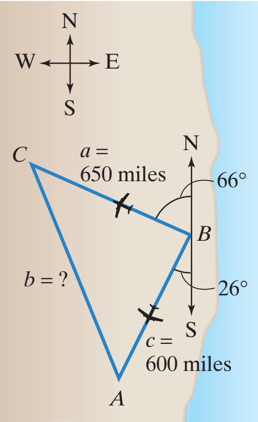


Figure 6.18 (repeated)

Let b = the distance between the planes after two hours. We can use a north-south line to find angle B in triangle ABC. Thus,

$$B = 180° - 66° - 26° = 88°.$$

We now have $a = 650$, $c = 600$, and $B = 88°$. We use the Law of Cosines to find b in this SAS situation.

$$b^2 = a^2 + c^2 - 2ac \cos B$$ Apply the Law of Cosines.

$$b^2 = 650^2 + 600^2 - 2(650)(600) \cos 88°$$ Substitute: $a = 650$, $c = 600$, and $B = 88°$.

$$\approx 755{,}278$$ Use a calculator.

$$b \approx \sqrt{755{,}278} \approx 869$$ Take the square root and solve for b.

After two hours, the planes are approximately 869 miles apart.

Check Point 3 Two airplanes leave an airport at the same time on different runways. One flies directly north at 400 miles per hour. The other airplane flies on a bearing of N75°E at 350 miles per hour. How far apart will the airplanes be after two hours?

3 Use Heron's formula to find the area of a triangle.

Heron's Formula

Approximately 2000 years ago, the Greek mathematician Heron of Alexandria derived a formula for the area of a triangle in terms of the lengths of its sides. A more modern derivation uses the Law of Cosines and can be found in the appendix.

Heron's Formula for the Area of a Triangle

The area of a triangle with sides a, b, and c is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is one-half its perimeter: $s = \frac{1}{2}(a + b + c)$.

EXAMPLE 4 Using Heron's Formula

Find the area of the triangle with $a = 12$ yards, $b = 16$ yards, and $c = 24$ yards. Round to the nearest square yard.

Solution Begin by calculating one-half the perimeter:

$$s = \frac{1}{2}(a + b + c) = \frac{1}{2}(12 + 16 + 24) = 26.$$

Use Heron's formula to find the area:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{26(26-12)(26-16)(26-24)}$$
$$= \sqrt{7280} \approx 85.$$

The area of the triangle is approximately 85 square yards.

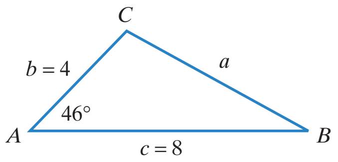
Check Point 4 Find the area of the triangle with $a = 6$ meters, $b = 16$ meters, and $c = 18$ meters. Round to the nearest square meter.

Exercise Set 6.2

Practice Exercises

In Exercises 1–8, solve each triangle. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

1.



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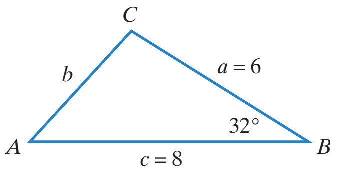

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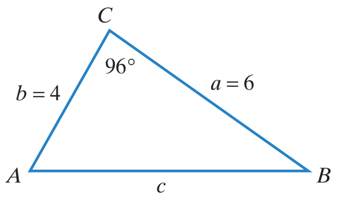

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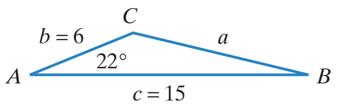

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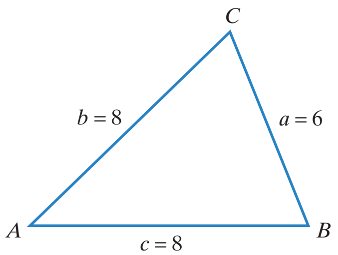

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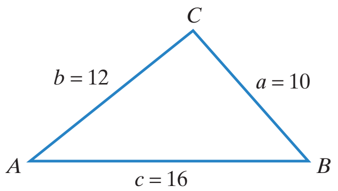

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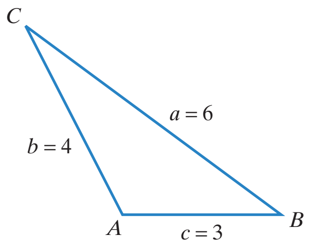

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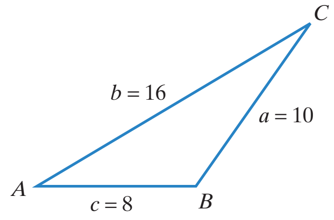


In Exercises 9–24, solve each triangle. Round lengths to the nearest tenth and angle measures to the nearest degree.

9. $a = 5, b = 7, C = 42°$

10. $a = 10, b = 3, C = 15°$

11. $b = 5, c = 3, A = 102°$

12. $b = 4, c = 1, A = 100°$

13. $a = 6, c = 5, B = 50°$

14. $a = 4, c = 7, B = 55°$

15. $a = 5, c = 2, B = 90°$

16. $a = 7, c = 3, B = 90°$

17. $a = 5, b = 7, c = 10$

18. $a = 4, b = 6, c = 9$

19. $a = 3, b = 9, c = 8$

20. $a = 4, b = 7, c = 6$

21. $a = 3, b = 3, c = 3$

22. $a = 5, b = 5, c = 5$

23. $a = 63, b = 22, c = 50$

24. $a = 66, b = 25, c = 45$

In Exercises 25–30, use Heron's formula to find the area of each triangle. Round to the nearest square unit.

25. $a = 4$ feet, $b = 4$ feet, $c = 2$ feet

26. $a = 5$ feet, $b = 5$ feet, $c = 4$ feet

27. $a = 14$ meters, $b = 12$ meters, $c = 4$ meters

28. $a = 16$ meters, $b = 10$ meters, $c = 8$ meters

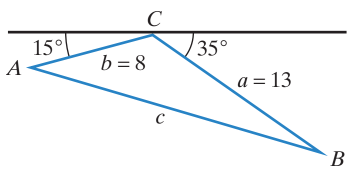
29. $a = 11$ yards, $b = 9$ yards, $c = 7$ yards

30. $a = 13$ yards, $b = 9$ yards, $c = 5$ yards

Practice Plus

In Exercises 31–32, solve each triangle. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

31.



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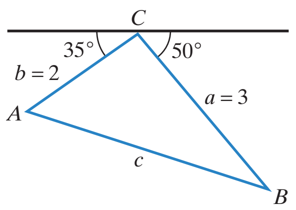
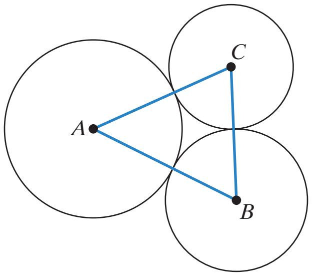


In Exercises 33–34, the three circles are arranged so that they touch each other, as shown in the figure. Use the given radii for the circles with centers A, B, and C, respectively, to solve triangle ABC. Round angle measures to the nearest degree.



33. 5.0, 4.0, 3.5

34. 7.5, 4.3, 3.0

In Exercises 35–36, the three given points are the vertices of a triangle. Solve each triangle, rounding lengths of sides to the nearest tenth and angle measures to the nearest degree.

35. $A(0, 0), B(-3, 4), C(3, -1)$

36. $A(0, 0), B(4, -3), C(1, -5)$

Application Exercises

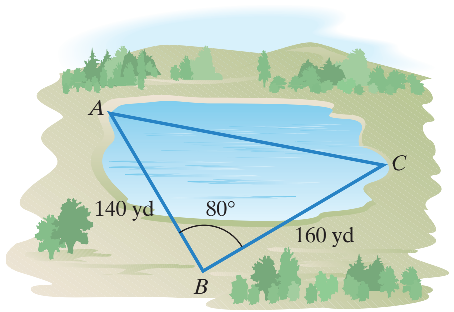
37. Use **Figure 6.13** on page 656 to find the pace angle, to the nearest degree, for the carnivore. Does the angle indicate that this dinosaur was an efficient walker? Describe your answer.

38. Use **Figure 6.13** on page 656 to find the pace angle, to the nearest degree, for the herbivore. Does the angle indicate that this dinosaur was an efficient walker? Describe your answer.

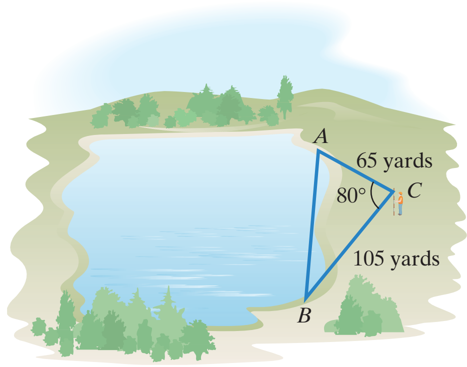
39. Two ships leave a harbor at the same time. One ship travels on a bearing of S12°W at 14 miles per hour. The other ship travels on a bearing of N75°E at 10 miles per hour. How far apart will the ships be after three hours? Round to the nearest tenth of a mile.

40. A plane leaves airport A and travels 580 miles to airport B on a bearing of N34°E. The plane later leaves airport B and travels to airport C 400 miles away on a bearing of S74°E. Find the distance from airport A to airport C to the nearest tenth of a mile.

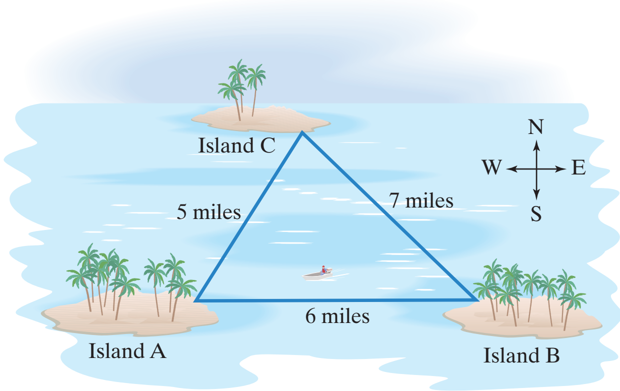
41. Find the distance across the lake from A to C, to the nearest yard, using the measurements shown in the figure.



42. To find the distance across a protected cove at a lake, a surveyor makes the measurements shown in the figure. Use these measurements to find the distance from A to B to the nearest yard.



The diagram shows three islands in Florida Bay. You rent a boat and plan to visit each of these remote islands. Use the diagram to solve Exercises 43–44.



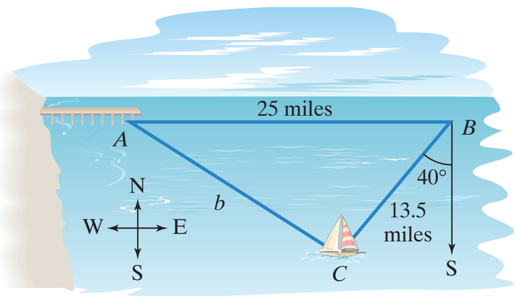
43. If you are on island A, on what bearing should you navigate to go to island C?

44. If you are on island B, on what bearing should you navigate to go to island C?

45. You are on a fishing boat that leaves its pier and heads east. After traveling for 25 miles, there is a report warning of rough seas directly south. The captain turns the boat and follows a bearing of S40°W for 13.5 miles.

a. At this time, how far are you from the boat's pier? Round to the nearest tenth of a mile.

b. What bearing could the boat have originally taken to arrive at this spot?

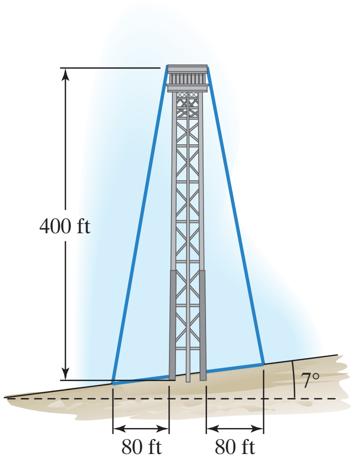


46. You are on a fishing boat that leaves its pier and heads east. After traveling for 30 miles, there is a report warning of rough seas directly south. The captain turns the boat and follows a bearing of S45°W for 12 miles.

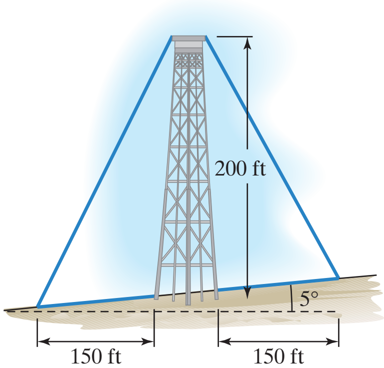
a. At this time, how far are you from the boat's pier? Round to the nearest tenth of a mile.

b. What bearing could the boat have originally taken to arrive at this spot?

47. The figure shows a 400-foot tower on the side of a hill that forms a 7° angle with the horizontal. Find the length of each of the two guy wires that are anchored 80 feet uphill and downhill from the tower's base and extend to the top of the tower. Round to the nearest tenth of a foot.



48. The figure shows a 200-foot tower on the side of a hill that forms a 5° angle with the horizontal. Find the length of each of the two guy wires that are anchored 150 feet uphill and downhill from the tower's base and extend to the top of the tower. Round to the nearest tenth of a foot.



49. A Major League baseball diamond has four bases forming a square whose sides measure 90 feet each. The pitcher's mound is 60.5 feet from home plate on a line joining home plate and second base. Find the distance from the pitcher's mound to first base. Round to the nearest tenth of a foot.

50. A Little League baseball diamond has four bases forming a square whose sides measure 60 feet each. The pitcher's mound is 46 feet from home plate on a line joining home plate and second base. Find the distance from the pitcher's mound to third base. Round to the nearest tenth of a foot.

51. A piece of commercial real estate is priced at \$3.50 per square foot. Find the cost, to the nearest dollar, of a triangular lot measuring 240 feet by 300 feet by 420 feet.

52. A piece of commercial real estate is priced at \$4.50 per square foot. Find the cost, to the nearest dollar, of a triangular lot measuring 320 feet by 510 feet by 410 feet.

Writing in Mathematics

53. Without using symbols, state the Law of Cosines in your own words.

54. Why can't the Law of Sines be used in the first step to solve an SAS triangle?

55. Describe a strategy for solving an SAS triangle.

56. Describe a strategy for solving an SSS triangle.

57. Under what conditions would you use Heron's formula to find the area of a triangle?

58. Describe an applied problem that can be solved using the Law of Cosines, but not the Law of Sines.

59. The pitcher on a Little League team is studying angles in geometry and has a question. "Coach, suppose I'm on the pitcher's mound facing home plate. I catch a fly ball hit in my direction. If I turn to face first base and throw the ball, through how many degrees should I turn for a direct throw?" Use the information given in Exercise 50 and write an answer to the pitcher's question. Without getting too technical, describe to the pitcher how you obtained this angle.

Critical Thinking Exercises

Make Sense? *In Exercises 60–63, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

60. The Law of Cosines is similar to the Law of Sines, with all the sines replaced with cosines.

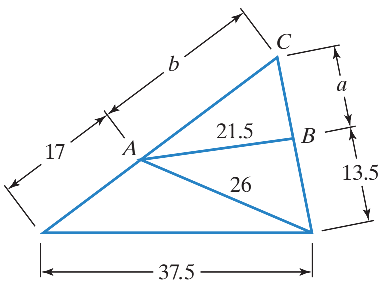
61. If I know the measures of all three angles of an oblique triangle, neither the Law of Sines nor the Law of Cosines can be used to find the length of a side.

62. I noticed that for a right triangle, the Law of Cosines reduces to the Pythagorean Theorem.

63. Solving an SSS triangle, I do not have to be concerned about the ambiguous case when using the Law of Sines.

64. The lengths of the diagonals of a parallelogram are 20 inches and 30 inches. The diagonals intersect at an angle of 35°. Find the lengths of the parallelogram's sides. (*Hint:* Diagonals of a parallelogram bisect one another.)

65. Use the figure to solve triangle ABC. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.



66. The minute hand and the hour hand of a clock have lengths m inches and h inches, respectively. Determine the distance between the tips of the hands at 10:00 in terms of m and h.

Group Exercise

67. The group should design five original problems that can be solved using the Laws of Sines and Cosines. At least two problems should be solved using the Law of Sines, one should be the ambiguous case, and at least two problems should be solved using the Law of Cosines. At least one problem should be an application problem using the Law of Sines and at least one problem should involve an application using the Law of Cosines. The group should turn in both the problems and their solutions.

Preview Exercises

Exercises 68–70 will help you prepare for the material covered in the next section.

68. Graph: $y = 3$.

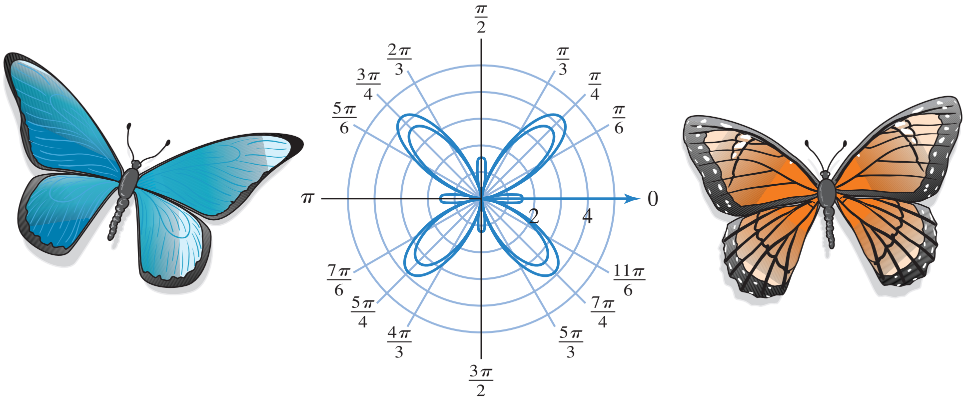
69. Graph: $x^2 + (y - 1)^2 = 1$.

70. Complete the square and write the equation in standard form: $x^2 + 6x + y^2 = 0$. Then give the center and radius of the circle, and graph the equation.

Section 6.3 Polar Coordinates

Objectives

1. Plot points in the polar coordinate system.
2. Find multiple sets of polar coordinates for a given point.
3. Convert a point from polar to rectangular coordinates.
4. Convert a point from rectangular to polar coordinates.
5. Convert an equation from rectangular to polar coordinates.
6. Convert an equation from polar to rectangular coordinates.



Butterflies are among the most celebrated of all insects. It's hard not to notice their beautiful colors and graceful flight. Their symmetry can be explored with trigonometric functions and a system for plotting points called the *polar coordinate system*. In many cases, polar coordinates are simpler and easier to use than rectangular coordinates.

1 Plot points in the polar coordinate system.

Plotting Points in the Polar Coordinate System

The foundation of the polar coordinate system is a horizontal ray that extends to the right. The ray is called the **polar axis** and is shown in **Figure 6.19**. The endpoint of the ray is called the **pole**.

A point P in the polar coordinate system is represented by an ordered pair of numbers (r, θ). **Figure 6.20** shows $P = (r, \theta)$ in the polar coordinate system.

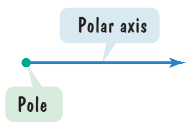


Figure 6.19

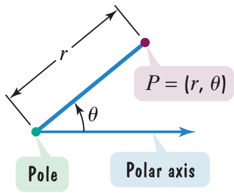


Figure 6.20 Representing a point in the polar coordinate system

- r is a directed distance from the pole to P. (We shall see that r can be positive, negative, or zero.)
- θ is an angle from the polar axis to the line segment from the pole to P. This angle can be measured in degrees or radians. Positive angles are measured counterclockwise from the polar axis. Negative angles are measured clockwise from the polar axis.

We refer to the ordered pair (r, θ) as the **polar coordinates** of P.