Section 9.2

The Hyperbola

Objectives

1. Locate a hyperbola's vertices and foci.
2. Write equations of hyperbolas in standard form.
3. Graph hyperbolas centered at the origin.
4. Graph hyperbolas not centered at the origin.
5. Solve applied problems involving hyperbolas.

St. Mary's Cathedral

Conic sections are often used to create unusual architectural designs. The top of St. Mary's Cathedral in San Francisco is a 2135-cubic-foot dome with walls rising 200 feet above the floor and supported by four massive concrete pylons that extend 94 feet into the ground. Cross sections of the roof are parabolas and hyperbolas. In this section, we study the curve with two parts known as the hyperbola.

Definition of a Hyperbola

Figure 9.13 shows a cylindrical lampshade casting two shadows on a wall. These shadows indicate the distinguishing feature of hyperbolas: Their graphs contain two disjoint parts, called **branches**. Although each branch might look like a parabola, its shape is actually quite different.

Figure 9.13 Casting hyperbolic shadows

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the *sum* of the distances to the foci is a constant. By contrast, for a hyperbola the *difference* of the distances to the foci is a constant.

Definition of a Hyperbola

A **hyperbola** is the set of points in a plane the difference of whose distances from two fixed points, called foci, is constant.

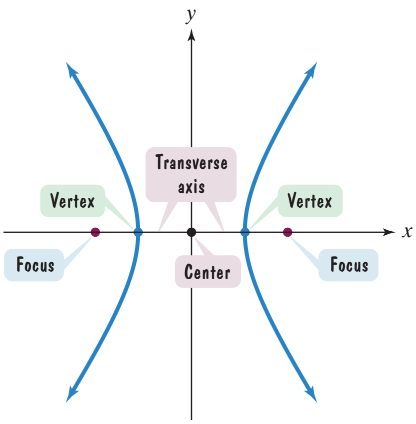


Figure 9.14 The two branches of a hyperbola

Figure 9.14 illustrates the two branches of a hyperbola. The line through the foci intersects the hyperbola at two points, called the **vertices**. The line segment that joins the vertices is the **transverse axis**. The midpoint of the transverse axis is the **center** of the hyperbola. Notice that the center lies midway between the vertices, as well as midway between the foci.

Standard Form of the Equation of a Hyperbola

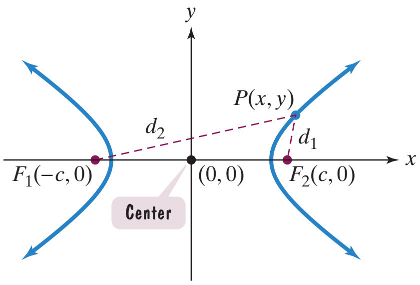


Figure 9.15

The rectangular coordinate system enables us to translate a hyperbola's geometric definition into an algebraic equation. **Figure 9.15** is our starting point for obtaining an equation. We place the foci, F_1 and F_2, on the x-axis at the points $(-c, 0)$ and $(c, 0)$. Note that the center of this hyperbola is at the origin. We let (x, y) represent the coordinates of any point, P, on the hyperbola.

What does the definition of a hyperbola tell us about the point (x, y) in **Figure 9.15**? For any point (x, y) on the hyperbola, the absolute value of the difference of the distances from the two foci, $|d_2 - d_1|$, must be constant. We denote this constant by $2a$, just as we did for the ellipse. Thus, the point (x, y) is on the hyperbola if and only if

$$|d_2 - d_1| = 2a.$$

$$\left|\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2}\right| = 2a \quad \text{Use the distance formula.}$$

After eliminating radicals and simplifying, we obtain

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2).$$

For convenience, let $b^2 = c^2 - a^2$. Substituting b^2 for $c^2 - a^2$ in the preceding equation, we obtain

$$b^2x^2 - a^2y^2 = a^2b^2.$$

$$\frac{b^2x^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \quad \text{Divide both sides by } a^2b^2.$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Simplify.}$$

This last equation is called the **standard form of the equation of a hyperbola centered at the origin**. There are two such equations. The first is for a hyperbola in which the transverse axis lies on the x-axis. The second is for a hyperbola in which the transverse axis lies on the y-axis.

Standard Forms of the Equations of a Hyperbola

The **standard form of the equation of a hyperbola** with center at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Figure 9.16(a) illustrates that for the equation on the left, the transverse axis lies on the x-axis. **Figure 9.16(b)** illustrates that for the equation on the right, the transverse axis lies on the y-axis. The vertices are a units from the center and the foci are c units from the center. For both equations, $b^2 = c^2 - a^2$. Equivalently, $c^2 = a^2 + b^2$.

Study Tip

The form $c^2 = a^2 + b^2$ is the one you should remember. When finding the foci, this form is easy to manipulate.

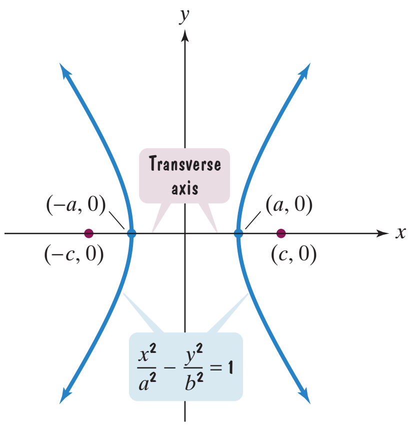


Figure 9.16(a) Transverse axis lies on the x-axis.

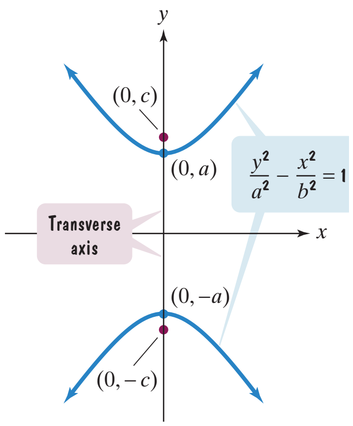


Figure 9.16(b) Transverse axis lies on the y-axis.

Study Tip

When the x^2-term is preceded by a plus sign, the transverse axis is horizontal. When the y^2-term is preceded by a plus sign, the transverse axis is vertical.

1 Locate a hyperbola's vertices and foci.

Study Tip

Notice the sign difference between the following equations:
Finding an ellipse's foci:

$$c^2 = a^2 - b^2$$

Finding a hyperbola's foci:

$$c^2 = a^2 + b^2.$$

Using the Standard Form of the Equation of a Hyperbola

We can use the standard form of the equation of a hyperbola to find its vertices and locate its foci. Because the vertices are a units from the center, begin by identifying a^2 in the equation. In the standard form of a hyperbola's equation, **a^2 is the number under the variable whose term is preceded by a plus sign** $(+)$. If the x^2-term is preceded by a plus sign, the transverse axis lies along the x-axis. Thus, the vertices are a units to the left and right of the origin. If the y^2-term is preceded by a plus sign, the transverse axis lies along the y-axis. Thus, the vertices are a units above and below the origin.

We know that the foci are c units from the center. The substitution that is used to derive the hyperbola's equation, $c^2 = a^2 + b^2$, is needed to locate the foci when a^2 and b^2 are known.

EXAMPLE 1 Finding Vertices and Foci from a Hyperbola's Equation

Find the vertices and locate the foci for each of the following hyperbolas with the given equation:

a. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ **b.** $\frac{y^2}{9} - \frac{x^2}{16} = 1.$

Solution Both equations are in standard form. We begin by identifying a^2 and b^2 in each equation.

a. The first equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$a^2 = 16$. This is the denominator of the term preceded by a plus sign.

$b^2 = 9$. This is the denominator of the term preceded by a minus sign.

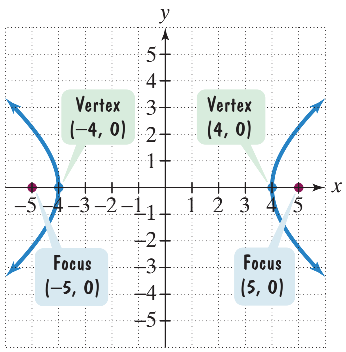


Figure 9.17 The graph of $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Because the x^2-term is preceded by a plus sign, the transverse axis lies along the x-axis. Thus, the vertices are a units to the *left* and *right* of the origin. Based on the standard form of the equation, we know the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 16$, $a = 4$. Thus, the vertices are $(-4, 0)$ and $(4, 0)$, shown in **Figure 9.17**.

We use $c^2 = a^2 + b^2$ to find the foci, which are located at $(-c, 0)$ and $(c, 0)$. We know that $a^2 = 16$ and $b^2 = 9$; we need to find c^2 in order to find c.

$$c^2 = a^2 + b^2 = 16 + 9 = 25$$

Because $c^2 = 25$, $c = 5$. The foci are located at $(-5, 0)$ and $(5, 0)$. They are shown in **Figure 9.17**.

b. The second given equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

$a^2 = 9$. This is the denominator of the term preceded by a plus sign.

$b^2 = 16$. This is the denominator of the term preceded by a minus sign.

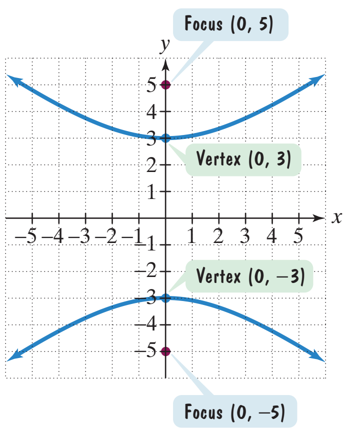


Figure 9.18 The graph of $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Because the y^2-term is preceded by a plus sign, the transverse axis lies along the y-axis. Thus, the vertices are a units *above* and *below* the origin. Based on the standard form of the equation, we know the vertices are $(0, -a)$ and $(0, a)$. Because $a^2 = 9$, $a = 3$. Thus, the vertices are $(0, -3)$ and $(0, 3)$, shown in **Figure 9.18**.

We use $c^2 = a^2 + b^2$ to find the foci, which are located at $(0, -c)$ and $(0, c)$.

$$c^2 = a^2 + b^2 = 9 + 16 = 25$$

Because $c^2 = 25$, $c = 5$. The foci are located at $(0, -5)$ and $(0, 5)$. They are shown in **Figure 9.18**.

Check Point 1 Find the vertices and locate the foci for each of the following hyperbolas with the given equation:

a. $\frac{x^2}{25} - \frac{y^2}{16} = 1$ **b.** $\frac{y^2}{25} - \frac{x^2}{16} = 1.$

In Example 1, we used equations of hyperbolas to find their foci and vertices. In the next example, we reverse this procedure.

2 Write equations of hyperbolas in standard form.

EXAMPLE 2 Finding the Equation of a Hyperbola from Its Foci and Vertices

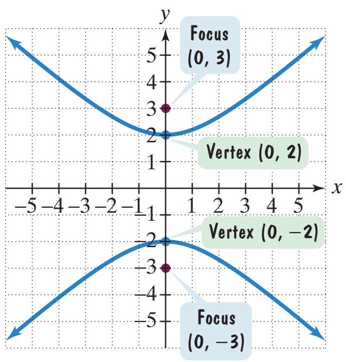


Figure 9.19

Find the standard form of the equation of a hyperbola with foci at $(0, -3)$ and $(0, 3)$ and vertices $(0, -2)$ and $(0, 2)$, shown in **Figure 9.19**.

Solution Because the foci are located at $(0, -3)$ and $(0, 3)$, on the y-axis, the transverse axis lies on the y-axis. The center of the hyperbola is midway between the foci, located at $(0, 0)$. Thus, the form of the equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

We need to determine the values for a^2 and b^2. The distance from the center, $(0, 0)$, to either vertex, $(0, -2)$ or $(0, 2)$, is 2, so $a = 2$.

$$\frac{y^2}{2^2} - \frac{x^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{4} - \frac{x^2}{b^2} = 1$$

We must still find b^2. The distance from the center, $(0, 0)$, to either focus, $(0, -3)$ or $(0, 3)$, is 3. Thus, $c = 3$. Using $c^2 = a^2 + b^2$, we have

$$3^2 = 2^2 + b^2$$

and

$$b^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Substituting 5 for b^2 in $\frac{y^2}{4} - \frac{x^2}{b^2} = 1$ gives us the standard form of the hyperbola's equation. The equation is

$$\frac{y^2}{4} - \frac{x^2}{5} = 1.$$

Check Point 2 Find the standard form of the equation of a hyperbola with foci at $(0, -5)$ and $(0, 5)$ and vertices $(0, -3)$ and $(0, 3)$.

The Asymptotes of a Hyperbola

As x and y get larger, the two branches of the graph of a hyperbola approach a pair of intersecting straight lines, called **asymptotes**. The asymptotes pass through the center of the hyperbola and are helpful in graphing hyperbolas.

Figure 9.20 shows the asymptotes for the graphs of hyperbolas centered at the origin. The asymptotes pass through the corners of a rectangle. Note that the dimensions of this rectangle are $2a$ by $2b$. The line segment of length $2b$ is the **conjugate axis** of the hyperbola and is perpendicular to the transverse axis through the center of the hyperbola.

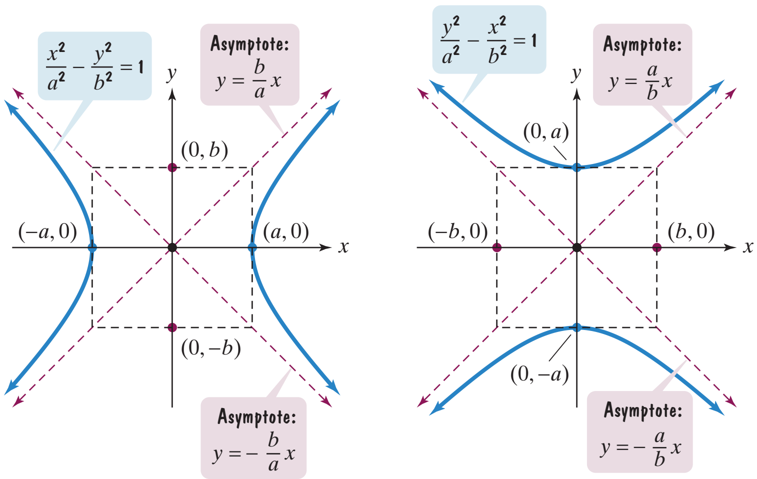


Figure 9.20 Asymptotes of a hyperbola

The Asymptotes of a Hyperbola Centered at the Origin

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has a horizontal transverse axis and two asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has a vertical transverse axis and two asymptotes

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.$$

Why are $y = \pm\frac{b}{a}x$ the asymptotes for a hyperbola whose transverse axis is horizontal? The proof can be found in Appendix A.

3 Graph hyperbolas centered at the origin.

Graphing Hyperbolas Centered at the Origin

Hyperbolas are graphed using vertices and asymptotes.

Graphing Hyperbolas

1. Locate the vertices.
2. Use dashed lines to draw the rectangle centered at the origin with sides parallel to the axes, crossing one axis at $\pm a$ and the other at $\pm b$.
3. Use dashed lines to draw the diagonals of this rectangle and extend them to obtain the asymptotes.
4. Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.

EXAMPLE 3 Graphing a Hyperbola

Graph and locate the foci: $\frac{x^2}{25} - \frac{y^2}{16} = 1$. What are the equations of the asymptotes?

Solution

Step 1 Locate the vertices. The given equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with $a^2 = 25$ and $b^2 = 16$.

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

$a^2 = 25$ $b^2 = 16$

Based on the standard form of the equation with the transverse axis on the x-axis, we know that the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 25$, $a = 5$. Thus, the vertices are $(-5, 0)$ and $(5, 0)$, shown in **Figure 9.21**.

Step 2 Draw a rectangle. Because $a^2 = 25$ and $b^2 = 16$, $a = 5$ and $b = 4$. We construct a rectangle to find the asymptotes, using -5 and 5 on the x-axis (the vertices are located here) and -4 and 4 on the y-axis. The rectangle passes through these four points, shown using dashed lines in **Figure 9.21**.

Step 3 Draw extended diagonals for the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle, shown in **Figure 9.21**, to obtain the graph of the asymptotes. Based on the standard form of the hyperbola's equation, the equations for these asymptotes are

$$y = \pm\frac{b}{a}x \quad \text{or} \quad y = \pm\frac{4}{5}x.$$

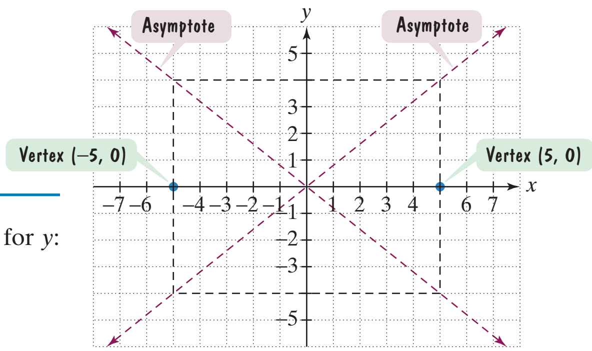


Figure 9.21 Preparing to graph $\frac{x^2}{25} - \frac{y^2}{16} = 1$

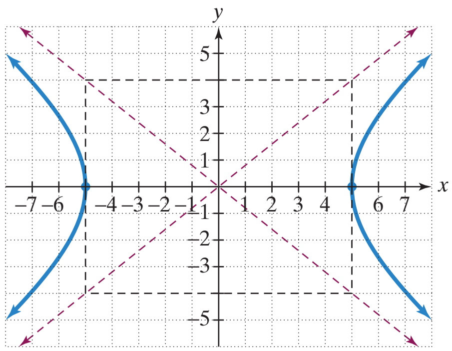


Figure 9.22 The graph of $\frac{x^2}{25} - \frac{y^2}{16} = 1$

Step 4 Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown in **Figure 9.22**.

We now consider the foci, located at $(-c, 0)$ and $(c, 0)$. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 25 + 16 = 41$$

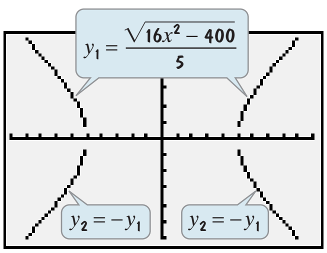
Because $c^2 = 41$, $c = \sqrt{41}$. The foci are located at $\left(-\sqrt{41}, 0\right)$ and $\left(\sqrt{41}, 0\right)$, approximately $(-6.4, 0)$ and $(6.4, 0)$.

Technology

Graph $\frac{x^2}{25} - \frac{y^2}{16} = 1$ by solving for y:

$$y_1 = \frac{\sqrt{16x^2 - 400}}{5}$$

$$y_2 = -\frac{\sqrt{16x^2 - 400}}{5} = -y_1.$$



[−10, 10, 1] by [−6, 6, 1]

Check Point 3 Graph and locate the foci: $\frac{x^2}{36} - \frac{y^2}{9} = 1$. What are the equations of the asymptotes?

EXAMPLE 4 Graphing a Hyperbola

Graph and locate the foci: $9y^2 - 4x^2 = 36$. What are the equations of the asymptotes?

Solution We begin by writing the equation in standard form. The right side should be 1, so we divide both sides by 36.

$$\frac{9y^2}{36} - \frac{4x^2}{36} = \frac{36}{36}$$

$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \quad \text{Simplify. The right side is now 1.}$$

Now we are ready to use our four-step procedure for graphing hyperbolas.

Step 1 Locate the vertices. The equation that we obtained is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, with $a^2 = 4$ and $b^2 = 9$.

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

$a^2 = 4$ $\qquad$ $b^2 = 9$

Based on the standard form of the equation with the transverse axis on the y-axis, we know that the vertices are $(0, -a)$ and $(0, a)$. Because $a^2 = 4$, $a = 2$. Thus, the vertices are $(0, -2)$ and $(0, 2)$, shown in **Figure 9.23**.

Step 2 Draw a rectangle. Because $a^2 = 4$ and $b^2 = 9$, $a = 2$ and $b = 3$. We construct a rectangle to find the asymptotes, using -2 and 2 on the y-axis (the vertices are located here) and -3 and 3 on the x-axis. The rectangle passes through these four points, shown using dashed lines in **Figure 9.23**.

Step 3 Draw extended diagonals of the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle, shown in **Figure 9.23**, to obtain the graph of the asymptotes. Based on the standard form of the hyperbola's equation, the equations of these asymptotes are

$$y = \pm\frac{a}{b}x \quad \text{or} \quad y = \pm\frac{2}{3}x.$$

Step 4 Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown in **Figure 9.24**.

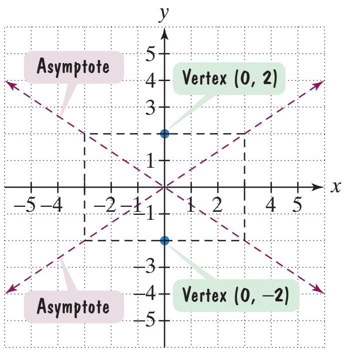


Figure 9.23 Preparing to graph $\frac{y^2}{4} - \frac{x^2}{9} = 1$

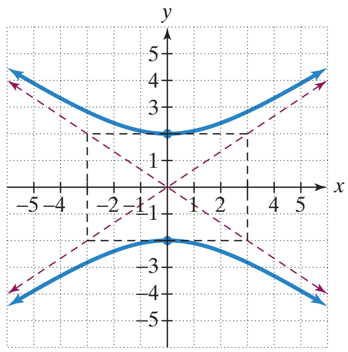


Figure 9.24 The graph of $\frac{y^2}{4} - \frac{x^2}{9} = 1$

We now consider the foci, located at $(0, -c)$ and $(0, c)$. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 4 + 9 = 13$$

Because $c^2 = 13$, $c = \sqrt{13}$. The foci are located at $(0, -\sqrt{13})$ and $(0, \sqrt{13})$, approximately $(0, -3.6)$ and $(0, 3.6)$.

Check Point 4 Graph and locate the foci: $y^2 - 4x^2 = 4$. What are the equations of the asymptotes?

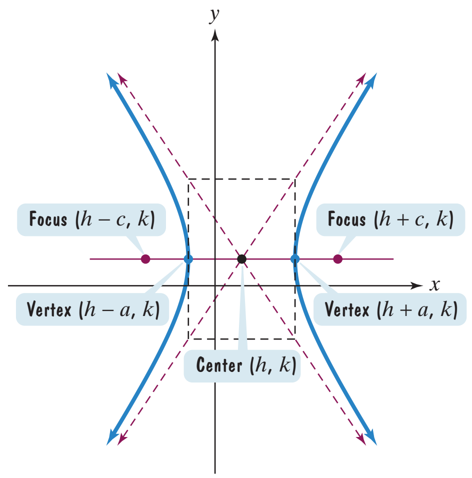
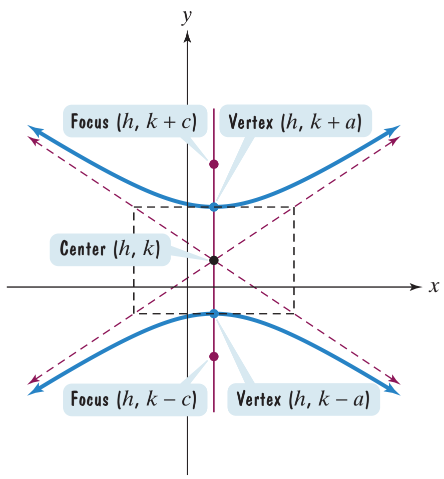
4 Graph hyperbolas not centered at the origin.

Translations of Hyperbolas

The graph of a hyperbola can be centered at (h, k), rather than at the origin. Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the hyperbola's equation.

Table 9.2 gives the standard forms of equations of hyperbolas centered at (h, k) and shows their graphs.

Table 9.2 Standard Forms of Equations of Hyperbolas Centered at (*h*, *k*)

Equation	Center	Transverse Axis	Vertices	Graph
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Vertices are a units right and a units left of center. Foci are c units right and c units left of center, where $c^2 = a^2 + b^2$.	(h, k)	Parallel to the x-axis; horizontal	$(h - a, k)$ $(h + a, k)$	
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ Vertices are a units above and a units below the center. Foci are c units above and c units below the center, where $c^2 = a^2 + b^2$.	(h, k)	Parallel to the y-axis; vertical	$(h, k - a)$ $(h, k + a)$	

EXAMPLE 5 **Graphing a Hyperbola Centered at (*h*, *k*)**

Graph: $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$. Where are the foci located? What are the equations of the asymptotes?

Solution In order to graph the hyperbola, we need to know its center, (h, k). In the standard forms of equations centered at (h, k), h is the number subtracted from x and k is the number subtracted from y.

This is $(x - h)^2$, with $h = 2$. This is $(y - k)^2$, with $k = 3$.

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

We see that $h = 2$ and $k = 3$. Thus, the center of the hyperbola, (h, k), is $(2, 3)$. We can graph the hyperbola by using vertices, asymptotes, and our four-step graphing procedure.

Step 1 Locate the vertices. To do this, we must identify a^2.

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

The form of this equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

$a^2 = 16$ $\quad$ $b^2 = 9$

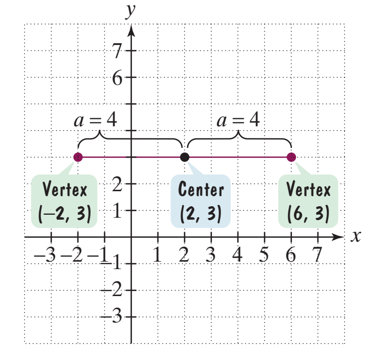


Figure 9.25 Locating a hyperbola's center and vertices

Based on the standard form of the equation with a horizontal transverse axis, the vertices are a units to the left and right of the center. Because $a^2 = 16$, $a = 4$. This means that the vertices are 4 units to the left and right of the center, $(2, 3)$. Four units to the left of $(2, 3)$ puts one vertex at $(2 - 4, 3)$, or $(-2, 3)$. Four units to the right of $(2, 3)$ puts the other vertex at $(2 + 4, 3)$, or $(6, 3)$. The vertices are shown in **Figure 9.25**.

Step 2 Draw a rectangle. Because $a^2 = 16$ and $b^2 = 9$, $a = 4$ and $b = 3$. The rectangle passes through points that are 4 units to the right and left of the center (the vertices are located here) and 3 units above and below the center. The rectangle is shown using dashed lines in **Figure 9.26**.

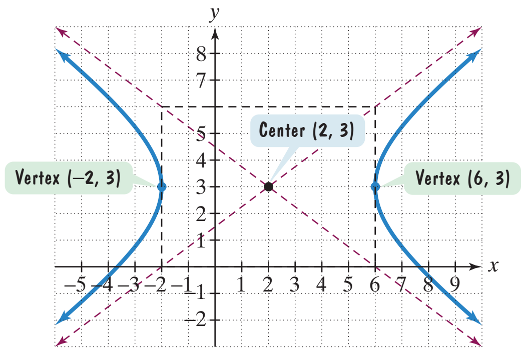


Figure 9.26 The graph of $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$

Step 3 Draw extended diagonals of the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle, shown in **Figure 9.26**, to obtain the graph of the asymptotes. The equations of the asymptotes of the unshifted hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are $y = \pm\frac{b}{a}x$, or $y = \pm\frac{3}{4}x$. Thus, the asymptotes for the hyperbola that is shifted two units to the right and three units up, namely

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

have equations that can be expressed as

$$y - 3 = \pm\frac{3}{4}(x - 2).$$

Step 4 Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown in **Figure 9.26**.

We now consider the foci, located c units to the right and left of the center. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 16 + 9 = 25$$

Because $c^2 = 25$, $c = 5$. This means that the foci are 5 units to the left and right of the center, $(2, 3)$. Five units to the left of $(2, 3)$ puts one focus at $(2 - 5, 3)$, or $(-3, 3)$. Five units to the right of $(2, 3)$ puts the other focus at $(2 + 5, 3)$, or $(7, 3)$.

Study Tip

You can also use the point-slope form of a line's equation

$$y - y_1 = m(x - x_1)$$

to find the equations of the asymptotes. The center of the hyperbola, (h, k), is a point on each asymptote, so $x_1 = h$ and $y_1 = k$. The slopes, m, are $\pm\frac{b}{a}$ for a horizontal transverse axis and $\pm\frac{a}{b}$ for a vertical transverse axis.

Check Point 5 Graph: $\frac{(x-3)^2}{4} - \frac{(y-1)^2}{1} = 1$. Where are the foci located? What are the equations of the asymptotes?

In our next example, it is necessary to convert the equation of a hyperbola to standard form by completing the square on x and y.

EXAMPLE 6 Graphing a Hyperbola Centered at (h, k)

Graph: $4x^2 - 24x - 25y^2 + 250y - 489 = 0$. Where are the foci located? What are the equations of the asymptotes?

Solution We begin by completing the square on x and y.

$$4x^2 - 24x - 25y^2 + 250y - 489 = 0$$
This is the given equation.

$$(4x^2 - 24x) + (-25y^2 + 250y) = 489$$
Group terms and add 489 to both sides.

$$4(x^2 - 6x + \square) - 25(y^2 - 10y + \square) = 489$$
Factor out 4 and −25, respectively, so coefficients of x^2 and y^2 are 1.

$$4(x^2 - 6x + 9) - 25(y^2 - 10y + 25) = 489 + 36 + (-625)$$
Complete each square by adding the square of half the coefficient of x and y, respectively.

We added $4 \cdot 9$, or 36, to the left side. We added $-25 \cdot 25$, or −625, to the left side. Add $36 + (-625)$ to the right side.

$$4(x-3)^2 - 25(y-5)^2 = -100$$
Factor.

$$\frac{4(x-3)^2}{-100} - \frac{25(y-5)^2}{-100} = \frac{-100}{-100}$$
Divide both sides by −100.

$$\frac{(x-3)^2}{-25} + \frac{(y-5)^2}{4} = 1$$
Simplify.

$$\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1$$
Write the equation in standard form, $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

This is $(y-k)^2$, with $k = 5$. This is $(x-h)^2$, with $h = 3$.

Study Tip

The hyperbola's center is (3, 5) because the last equation shows that 3 is subtracted from x and 5 is subtracted from y. Many students tend to read the equation from left to right and get the center backward. The hyperbola's center is *not* (5, 3).

We see that $h = 3$ and $k = 5$. Thus, the center of the hyperbola, (h, k), is (3, 5). Because the x^2-term is being subtracted, the transverse axis is vertical and the hyperbola opens upward and downward.

We use our four-step procedure to obtain the graph of

$$\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1.$$

$a^2 = 4$ $b^2 = 25$

Step 1 Locate the vertices. Based on the standard form of the equation with a vertical transverse axis, the vertices are a units above and below the center. Because $a^2 = 4$, $a = 2$. This means that the vertices are 2 units above and below the center, (3, 5). This puts the vertices at (3, 7) and (3, 3), shown in **Figure 9.27**.

Step 2 Draw a rectangle. Because $a^2 = 4$ and $b^2 = 25$, $a = 2$ and $b = 5$. The rectangle passes through points that are 2 units above and below the center (the vertices are located here) and 5 units to the right and left of the center. The rectangle is shown using dashed lines in **Figure 9.27**.

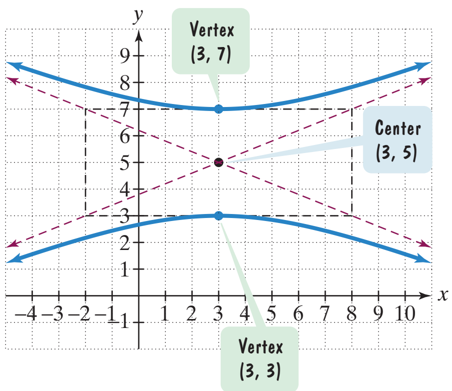


Figure 9.27 The graph of $\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1$

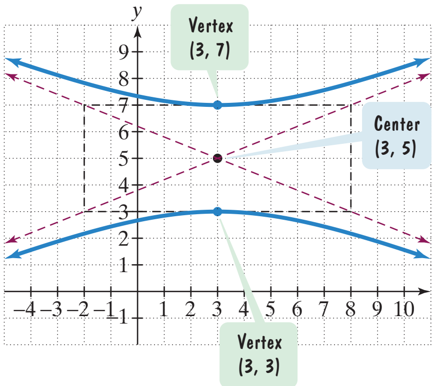


Figure 9.27 (repeated) The graph of $\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1$

Step 3 Draw extended diagonals of the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle, shown in **Figure 9.27**, to obtain the graph of the asymptotes. The equations of the asymptotes of the unshifted hyperbola $\frac{y^2}{4} - \frac{x^2}{25} = 1$ are $y = \pm\frac{a}{b}x$, or $y = \pm\frac{2}{5}x$. Thus, the asymptotes for the hyperbola that is shifted three units to the right and five units up, namely

$$\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1$$

have equations that can be expressed as

$$y - 5 = \pm\frac{2}{5}(x - 3).$$

Step 4 Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown in **Figure 9.27**.

We now consider the foci, located c units above and below the center, (3, 5). We find c using $c^2 = a^2 + b^2$.

$$c^2 = 4 + 25 = 29$$

Because $c^2 = 29$, $c = \sqrt{29}$. The foci are located at $\left(3, 5 + \sqrt{29}\right)$ and $\left(3, 5 - \sqrt{29}\right)$.

Check Point 6 Graph: $4x^2 - 24x - 9y^2 - 90y - 153 = 0$. Where are the foci located? What are the equations of the asymptotes?

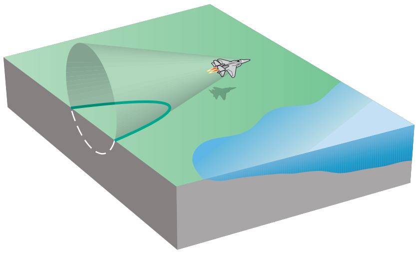
5 Solve applied problems involving hyperbolas.

Applications

Hyperbolas have many applications. When a jet flies at a speed greater than the speed of sound, the shock wave that is created is heard as a sonic boom. The wave has the shape of a cone. The shape formed as the cone hits the ground is one branch of a hyperbola.

Halley's Comet, a permanent part of our solar system, travels around the sun in an elliptical orbit. Other comets pass through the solar system only once, following a hyperbolic path with the sun as a focus.

Hyperbolas are of practical importance in fields ranging from architecture to navigation. Cooling towers used in the design for nuclear power plants have cross sections that are both ellipses and hyperbolas. Three-dimensional solids whose cross sections are hyperbolas are used in some rather unique architectural creations, including the TWA building at Kennedy Airport in New York City and the St. Louis Science Center Planetarium.



The hyperbolic shape of a sonic boom

EXAMPLE 7 An Application Involving Hyperbolas

An explosion is recorded by two microphones that are 2 miles apart. Microphone M_1 received the sound 4 seconds before microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.

Solution We begin by putting the microphones in a coordinate system. Because 1 mile = 5280 feet, we place M_1 5280 feet on a horizontal axis to the right of the origin and M_2 5280 feet on a horizontal axis to the left of the origin. **Figure 9.28** illustrates that the two microphones are 2 miles apart.

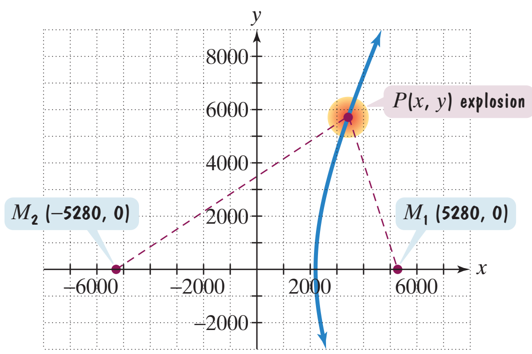
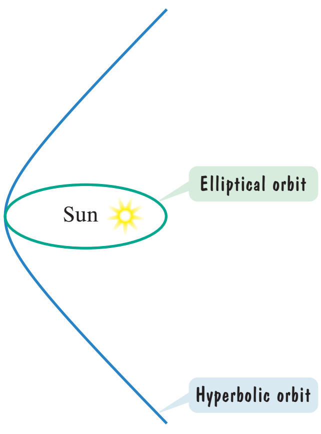


Figure 9.28 Locating an explosion on the branch of a hyperbola



Orbits of comets

We know that M_2 received the sound 4 seconds after M_1. Because sound travels at 1100 feet per second, the difference between the distance from P to M_1 and the distance from P to M_2 is 4400 feet. The set of all points P (or locations of the explosion) satisfying these conditions fits the definition of a hyperbola, with microphones M_1 and M_2 at the foci.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ Use the standard form of the hyperbola's equation. $P(x, y)$, the explosion point, lies on this hyperbola. We must find a^2 and b^2.

The difference between the distances, represented by $2a$ in the derivation of the hyperbola's equation, is 4400 feet. Thus, $2a = 4400$ and $a = 2200$.

$$\frac{x^2}{(2200)^2} - \frac{y^2}{b^2} = 1$$ Substitute 2200 for a.

$$\frac{x^2}{4{,}840{,}000} - \frac{y^2}{b^2} = 1$$ Square 2200.

We must still find b^2. We know that $a = 2200$. The distance from the center, $(0, 0)$, to either focus, $(-5280, 0)$ or $(5280, 0)$, is 5280. Thus, $c = 5280$. Using $c^2 = a^2 + b^2$, we have

$$5280^2 = 2200^2 + b^2$$

and

$$b^2 = 5280^2 - 2200^2 = 23{,}038{,}400.$$

The equation of the hyperbola with a microphone at each focus is

$$\frac{x^2}{4{,}840{,}000} - \frac{y^2}{23{,}038{,}400} = 1.$$ Substitute 23,038,400 for b^2.

We can conclude that the explosion occurred somewhere on the right branch (the branch closer to M_1) of the hyperbola given by this equation.

In Example 7, we determined that the explosion occurred somewhere along one branch of a hyperbola, but not exactly where on the hyperbola. If, however, we had received the sound from another pair of microphones, we could locate the sound along a branch of another hyperbola. The exact location of the explosion would be the point where the two hyperbolas intersect.

Check Point 7 Rework Example 7 assuming microphone M_1 receives the sound 3 seconds before microphone M_2.

Exercise Set 9.2

Practice Exercises

In Exercises 1–4, find the vertices and locate the foci of each hyperbola with the given equation. Then match each equation to one of the graphs that are shown and labeled (a)–(d).

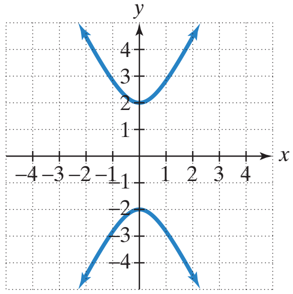
1. $\frac{x^2}{4} - \frac{y^2}{1} = 1$

2. $\frac{x^2}{1} - \frac{y^2}{4} = 1$

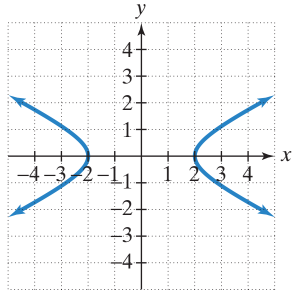
3. $\frac{y^2}{4} - \frac{x^2}{1} = 1$

4. $\frac{y^2}{1} - \frac{x^2}{4} = 1$

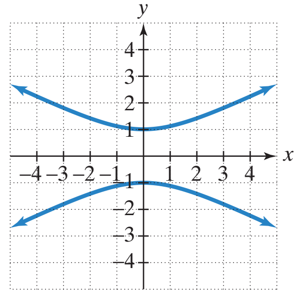
a.



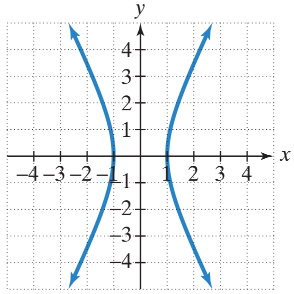
b.



c.



d.



In Exercises 5–12, find the standard form of the equation of each hyperbola satisfying the given conditions.

5. Foci: $(0, -3)$, $(0, 3)$; vertices: $(0, -1)$, $(0, 1)$

6. Foci: $(0, -6)$, $(0, 6)$; vertices: $(0, -2)$, $(0, 2)$

7. Foci: $(-4, 0)$, $(4, 0)$; vertices: $(-3, 0)$, $(3, 0)$

8. Foci: $(-7, 0)$, $(7, 0)$; vertices: $(-5, 0)$, $(5, 0)$

9. Endpoints of transverse axis: $(0, -6)$, $(0, 6)$; asymptote: $y = 2x$

10. Endpoints of transverse axis: $(-4, 0), (4, 0)$; asymptote: $y = 2x$

11. Center: $(4, -2)$; Focus: $(7, -2)$; vertex: $(6, -2)$

12. Center: $(-2, 1)$; Focus: $(-2, 6)$; vertex: $(-2, 4)$

In Exercises 13–26, use vertices and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

13. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

14. $\frac{x^2}{16} - \frac{y^2}{25} = 1$

15. $\frac{x^2}{100} - \frac{y^2}{64} = 1$

16. $\frac{x^2}{144} - \frac{y^2}{81} = 1$

17. $\frac{y^2}{16} - \frac{x^2}{36} = 1$

18. $\frac{y^2}{25} - \frac{x^2}{64} = 1$

19. $4y^2 - x^2 = 1$

20. $9y^2 - x^2 = 1$

21. $9x^2 - 4y^2 = 36$

22. $4x^2 - 25y^2 = 100$

23. $9y^2 - 25x^2 = 225$

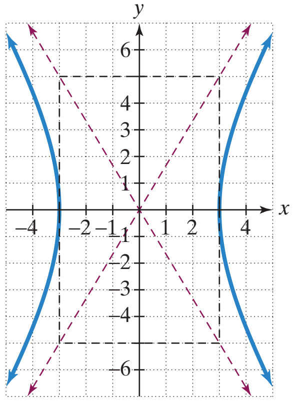
24. $16y^2 - 9x^2 = 144$

25. $y = \pm\sqrt{x^2 - 2}$

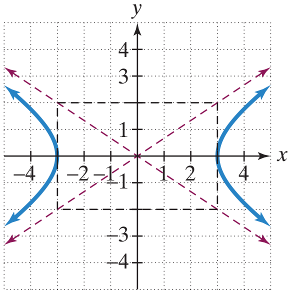
26. $y = \pm\sqrt{x^2 - 3}$

In Exercises 27–32, find the standard form of the equation of each hyperbola.

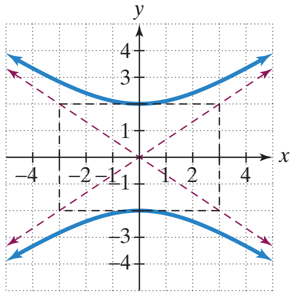
27.



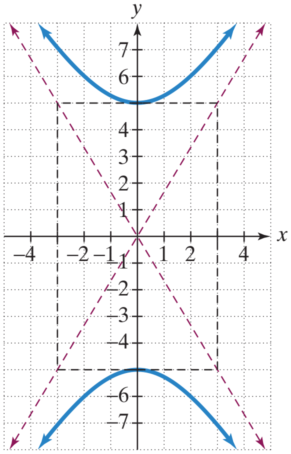
28.



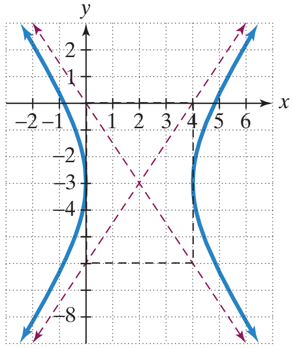
29.



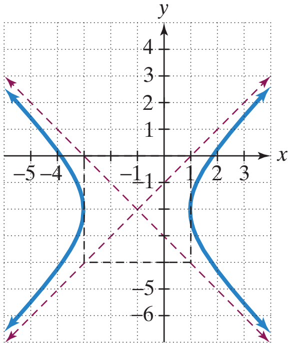
30.



31.



32.



In Exercises 33–42, use the center, vertices, and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

33. $\frac{(x+4)^2}{9} - \frac{(y+3)^2}{16} = 1$

34. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{25} = 1$

35. $\frac{(x+3)^2}{25} - \frac{y^2}{16} = 1$

36. $\frac{(x+2)^2}{9} - \frac{y^2}{25} = 1$

37. $\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1$

38. $\frac{(y-2)^2}{36} - \frac{(x+1)^2}{49} = 1$

39. $(x-3)^2 - 4(y+3)^2 = 4$

40. $(x+3)^2 - 9(y-4)^2 = 9$

41. $(x-1)^2 - (y-2)^2 = 3$

42. $(y-2)^2 - (x+3)^2 = 5$

In Exercises 43–50, convert each equation to standard form by completing the square on x and y. Then graph the hyperbola. Locate the foci and find the equations of the asymptotes.

43. $x^2 - y^2 - 2x - 4y - 4 = 0$

44. $4x^2 - y^2 + 32x + 6y + 39 = 0$

45. $16x^2 - y^2 + 64x - 2y + 67 = 0$

46. $9y^2 - 4x^2 - 18y + 24x - 63 = 0$

47. $4x^2 - 9y^2 - 16x + 54y - 101 = 0$

48. $4x^2 - 9y^2 + 8x - 18y - 6 = 0$

49. $4x^2 - 25y^2 - 32x + 164 = 0$

50. $9x^2 - 16y^2 - 36x - 64y + 116 = 0$

Practice Plus

In Exercises 51–56, graph each relation. Use the relation's graph to determine its domain and range.

51. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

52. $\frac{x^2}{25} - \frac{y^2}{4} = 1$

53. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

54. $\frac{x^2}{25} + \frac{y^2}{4} = 1$

55. $\frac{y^2}{16} - \frac{x^2}{9} = 1$

56. $\frac{y^2}{4} - \frac{x^2}{25} = 1$

In Exercises 57–60, find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

57. $\begin{cases} x^2 - y^2 = 4 \\ x^2 + y^2 = 4 \end{cases}$

58. $\begin{cases} x^2 - y^2 = 9 \\ x^2 + y^2 = 9 \end{cases}$

59. $\begin{cases} 9x^2 + y^2 = 9 \\ y^2 - 9x^2 = 9 \end{cases}$

60. $\begin{cases} 4x^2 + y^2 = 4 \\ y^2 - 4x^2 = 4 \end{cases}$

Application Exercises

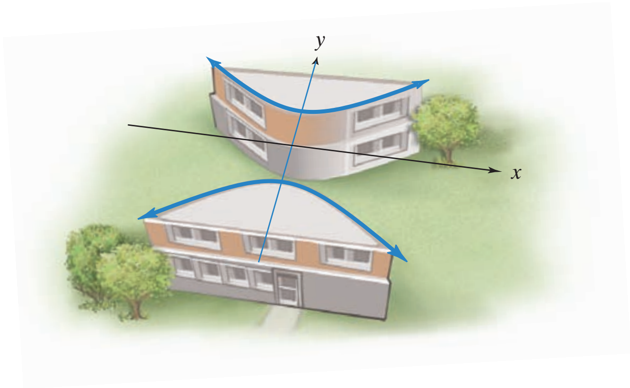
61. An explosion is recorded by two microphones that are 1 mile apart. Microphone M_1 received the sound 2 seconds before microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.

62. Radio towers A and B, 200 kilometers apart, are situated along the coast, with A located due west of B. Simultaneous radio signals are sent from each tower to a ship, with the signal from B received 500 microseconds before the signal from A.

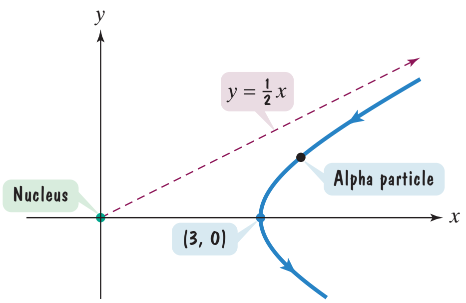
a. Assuming that the radio signals travel 300 meters per microsecond, determine the equation of the hyperbola on which the ship is located.

b. If the ship lies due north of tower B, how far out at sea is it?

63. An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250{,}000$, where x and y are in yards. How far apart are the houses at their closest point?



64. Scattering experiments, in which moving particles are deflected by various forces, led to the concept of the nucleus of an atom. In 1911, the physicist Ernest Rutherford (1871–1937) discovered that when alpha particles are directed toward the nuclei of gold atoms, they are eventually deflected along hyperbolic paths, illustrated in the figure. If a particle gets as close as 3 units to the nucleus along a hyperbolic path with an asymptote given by $y = \frac{1}{2}x$, what is the equation of its path?



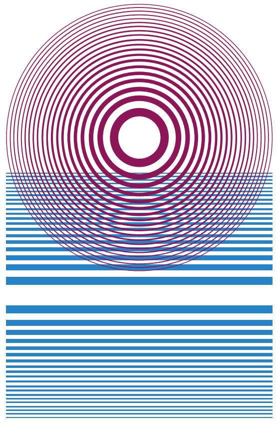
Moiré patterns, such as those shown in Exercises 65–66, can appear when two repetitive patterns overlap to produce a third, sometimes unintended, pattern.

a. *In each exercise, use the name of a conic section to describe the moiré pattern.*

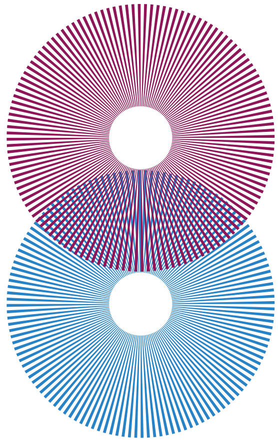
b. *Select one of the following equations that can possibly describe a conic section within the moiré pattern:*

$$x^2 + y^2 = 1; \quad x^2 - y^2 = 1; \quad x^2 + 4y^2 = 4.$$

65.



66.



Writing in Mathematics

67. What is a hyperbola?

68. Describe how to graph $\frac{x^2}{9} - \frac{y^2}{1} = 1$.

69. Describe how to locate the foci of the graph of $\frac{x^2}{9} - \frac{y^2}{1} = 1$.

70. Describe one similarity and one difference between the graphs of $\frac{x^2}{9} - \frac{y^2}{1} = 1$ and $\frac{y^2}{9} - \frac{x^2}{1} = 1$.

71. Describe one similarity and one difference between the graphs of $\frac{x^2}{9} - \frac{y^2}{1} = 1$ and $\frac{(x-3)^2}{9} - \frac{(y+3)^2}{1} = 1$.

72. How can you distinguish an ellipse from a hyperbola by looking at their equations?

73. In 1992, a NASA team began a project called Spaceguard Survey, calling for an international watch for comets that might collide with Earth. Why is it more difficult to detect a possible "doomsday comet" with a hyperbolic orbit than one with an elliptical orbit?

Technology Exercises

74. Use a graphing utility to graph any five of the hyperbolas that you graphed by hand in Exercises 13–26.

75. Use a graphing utility to graph any three of the hyperbolas that you graphed by hand in Exercises 33–42. First solve the given equation for y by using the square root property. Enter each of the two resulting equations to produce each branch of the hyperbola.

76. Use a graphing utility to graph any one of the hyperbolas that you graphed by hand in Exercises 43–50. Write the equation as a quadratic equation in y and use the quadratic formula to solve for y. Enter each of the two resulting equations to produce each branch of the hyperbola.

77. Use a graphing utility to graph $\frac{x^2}{4} - \frac{y^2}{9} = 0$. Is the graph a hyperbola? In general, what is the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$?

78. Graph $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ in the same viewing rectangle for values of a^2 and b^2 of your choice. Describe the relationship between the two graphs.

79. Write $4x^2 - 6xy + 2y^2 - 3x + 10y - 6 = 0$ as a quadratic equation in y and then use the quadratic formula to express y in terms of x. Graph the resulting two equations using a graphing utility in a $[-50, 70, 10]$ by $[-30, 50, 10]$ viewing rectangle. What effect does the xy-term have on the graph of the resulting hyperbola? What problems would you encounter if you attempted to write the given equation in standard form by completing the square?

80. Graph $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{x|x|}{16} - \frac{y|y|}{9} = 1$ in the same viewing rectangle. Explain why the graphs are not the same.

Critical Thinking Exercises

Make Sense? *In Exercises 81–84, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

81. I changed the addition in an ellipse's equation to subtraction and this changed its elongation from horizontal to vertical.

82. I noticed that the definition of a hyperbola closely resembles that of an ellipse in that it depends on the distances between a set of points in a plane to two fixed points, the foci.

83. I graphed a hyperbola centered at the origin that had y-intercepts, but no x-intercepts.

84. I graphed a hyperbola centered at the origin that was symmetric with respect to the x-axis and also symmetric with respect to the y-axis.

In Exercises 85–88, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

85. If one branch of a hyperbola is removed from a graph, then the branch that remains must define y as a function of x.

86. All points on the asymptotes of a hyperbola also satisfy the hyperbola's equation.

87. The graph of $\frac{x^2}{9} - \frac{y^2}{4} = 1$ does not intersect the line $y = -\frac{2}{3}x$.

88. Two different hyperbolas can never share the same asymptotes.

89. What happens to the shape of the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as $\frac{c}{a} \to \infty$, where $c^2 = a^2 + b^2$?

90. Find the standard form of the equation of the hyperbola with vertices $(5, -6)$ and $(5, 6)$, passing through $(0, 9)$.

91. Find the equation of a hyperbola whose asymptotes are perpendicular.

Preview Exercises

Exercises 92–94 will help you prepare for the material covered in the next section.

In Exercises 92–93, graph each parabola with the given equation.

92. $y = x^2 + 4x - 5$

93. $y = -3(x - 1)^2 + 2$

94. Isolate the terms involving y on the left side of the equation:

$$y^2 + 2y + 12x - 23 = 0.$$

Then write the equation in an equivalent form by completing the square on the left side.

Section 9.3 The Parabola

Objectives

1. Graph parabolas with vertices at the origin.
2. Write equations of parabolas in standard form.
3. Graph parabolas with vertices not at the origin.
4. Solve applied problems involving parabolas.

At first glance, this image looks like columns of smoke rising from a fire into a starry sky. Those are, indeed, stars in the background, but you are not looking at ordinary smoke columns. These stand almost 6 trillion miles high and are 7000 light-years from Earth—more than 400 million times as far away as the sun.

This NASA photograph is one of a series of stunning images captured from the ends of the universe by the Hubble Space Telescope. The image shows infant star systems the size of our solar system emerging from the gas and dust that shrouded their creation. Using a parabolic mirror that is 94.5 inches in diameter, the Hubble has provided answers to many of the profound mysteries of the cosmos: How big and how old is the universe? How did the galaxies come to exist? Do other Earth-like planets orbit other sun-like stars? In this section, we study parabolas and their applications, including parabolic shapes that gather distant rays of light and focus them into spectacular images.

Definition of a Parabola

In Chapter 2, we studied parabolas, viewing them as graphs of quadratic functions in the form

$$y = a(x - h)^2 + k \quad \text{or} \quad y = ax^2 + bx + c.$$