79. Write $4x^2 - 6xy + 2y^2 - 3x + 10y - 6 = 0$ as a quadratic equation in y and then use the quadratic formula to express y in terms of x. Graph the resulting two equations using a graphing utility in a $[-50, 70, 10]$ by $[-30, 50, 10]$ viewing rectangle. What effect does the xy-term have on the graph of the resulting hyperbola? What problems would you encounter if you attempted to write the given equation in standard form by completing the square?

80. Graph $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{x|x|}{16} - \frac{y|y|}{9} = 1$ in the same viewing rectangle. Explain why the graphs are not the same.

Critical Thinking Exercises

Make Sense? *In Exercises 81–84, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

81. I changed the addition in an ellipse's equation to subtraction and this changed its elongation from horizontal to vertical.

82. I noticed that the definition of a hyperbola closely resembles that of an ellipse in that it depends on the distances between a set of points in a plane to two fixed points, the foci.

83. I graphed a hyperbola centered at the origin that had y-intercepts, but no x-intercepts.

84. I graphed a hyperbola centered at the origin that was symmetric with respect to the x-axis and also symmetric with respect to the y-axis.

In Exercises 85–88, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

85. If one branch of a hyperbola is removed from a graph, then the branch that remains must define y as a function of x.

86. All points on the asymptotes of a hyperbola also satisfy the hyperbola's equation.

87. The graph of $\frac{x^2}{9} - \frac{y^2}{4} = 1$ does not intersect the line $y = -\frac{2}{3}x$.

88. Two different hyperbolas can never share the same asymptotes.

89. What happens to the shape of the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as $\frac{c}{a} \to \infty$, where $c^2 = a^2 + b^2$?

90. Find the standard form of the equation of the hyperbola with vertices $(5, -6)$ and $(5, 6)$, passing through $(0, 9)$.

91. Find the equation of a hyperbola whose asymptotes are perpendicular.

Preview Exercises

Exercises 92–94 will help you prepare for the material covered in the next section.

In Exercises 92–93, graph each parabola with the given equation.

92. $y = x^2 + 4x - 5$

93. $y = -3(x - 1)^2 + 2$

94. Isolate the terms involving y on the left side of the equation:

$$y^2 + 2y + 12x - 23 = 0.$$

Then write the equation in an equivalent form by completing the square on the left side.

Section 9.3 The Parabola

Objectives

1. Graph parabolas with vertices at the origin.
2. Write equations of parabolas in standard form.
3. Graph parabolas with vertices not at the origin.
4. Solve applied problems involving parabolas.

At first glance, this image looks like columns of smoke rising from a fire into a starry sky. Those are, indeed, stars in the background, but you are not looking at ordinary smoke columns. These stand almost 6 trillion miles high and are 7000 light-years from Earth—more than 400 million times as far away as the sun.

This NASA photograph is one of a series of stunning images captured from the ends of the universe by the Hubble Space Telescope. The image shows infant star systems the size of our solar system emerging from the gas and dust that shrouded their creation. Using a parabolic mirror that is 94.5 inches in diameter, the Hubble has provided answers to many of the profound mysteries of the cosmos: How big and how old is the universe? How did the galaxies come to exist? Do other Earth-like planets orbit other sun-like stars? In this section, we study parabolas and their applications, including parabolic shapes that gather distant rays of light and focus them into spectacular images.

Definition of a Parabola

In Chapter 2, we studied parabolas, viewing them as graphs of quadratic functions in the form

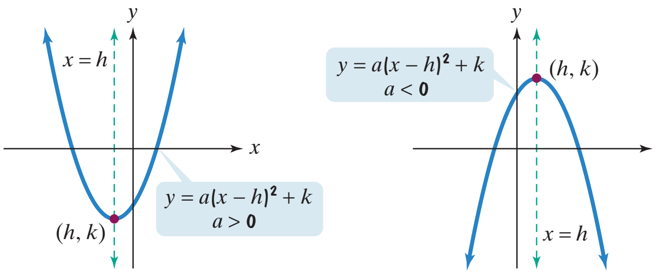
$$y = a(x - h)^2 + k \quad \text{or} \quad y = ax^2 + bx + c.$$

Study Tip

Here is a summary of what you should already know about graphing parabolas.

Graphing $y = a(x - h)^2 + k$ and $y = ax^2 + bx + c$

1. If $a > 0$, the graph opens upward. If $a < 0$, the graph opens downward.
2. The vertex of $y = a(x - h)^2 + k$ is (h, k).



3. The x-coordinate of the vertex of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

Parabolas can be given a geometric definition that enables us to include graphs that open to the left or to the right, as well as those that open obliquely. The definitions of ellipses and hyperbolas involved two fixed points, the foci. By contrast, the definition of a parabola is based on one point and a line.

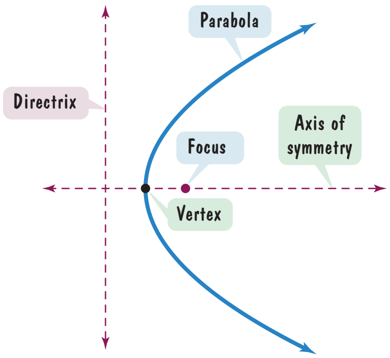


Figure 9.29

Definition of a Parabola

A **parabola** is the set of all points in a plane that are equidistant from a fixed line, the **directrix**, and a fixed point, the **focus**, that is not on the line (see **Figure 9.29**).

In **Figure 9.29**, find the line passing through the focus and perpendicular to the directrix. This is the **axis of symmetry** of the parabola. The point of intersection of the parabola with its axis of symmetry is called the **vertex**. Notice that the vertex is midway between the focus and the directrix.

Standard Form of the Equation of a Parabola

The rectangular coordinate system enables us to translate a parabola's geometric definition into an algebraic equation. **Figure 9.30** is our starting point for obtaining an equation. We place the focus on the x-axis at the point $(p, 0)$. The directrix has an equation given by $x = -p$. The vertex, located midway between the focus and the directrix, is at the origin.

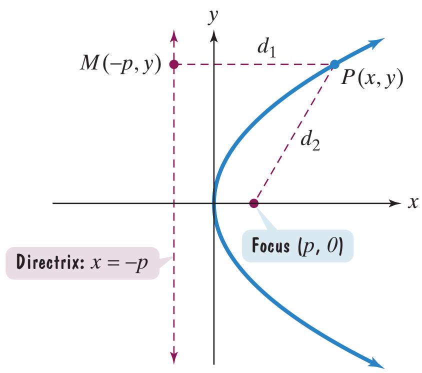


Figure 9.30

What does the definition of a parabola tell us about the point (x, y) in **Figure 9.30**? For any point (x, y) on the parabola, the distance d_1 to the directrix is equal to the distance d_2 to the focus. Thus, the point (x, y) is on the parabola if and only if

$$d_1 = d_2.$$

$$\sqrt{(x + p)^2 + (y - y)^2} = \sqrt{(x - p)^2 + (y - 0)^2} \quad \text{Use the distance formula.}$$

$$(x + p)^2 = (x - p)^2 + y^2 \quad \text{Square both sides of the equation.}$$

$$x^2 + 2px + p^2 = x^2 - 2px + p^2 + y^2$$ Square $x + p$ and $x - p$.

$$2px = -2px + y^2$$ Subtract $x^2 + p^2$ from both sides of the equation.

$$y^2 = 4px$$ Solve for y^2.

This last equation is called the **standard form of the equation of a parabola with its vertex at the origin**. There are two such equations, one for a focus on the x-axis and one for a focus on the y-axis.

Standard Forms of the Equations of a Parabola

The **standard form of the equation of a parabola** with vertex at the origin is

$$y^2 = 4px \quad \text{or} \quad x^2 = 4py.$$

Figure 9.31(a) illustrates that for the equation on the left, the focus is on the x-axis, which is the axis of symmetry. **Figure 9.31(b)** illustrates that for the equation on the right, the focus is on the y-axis, which is the axis of symmetry.

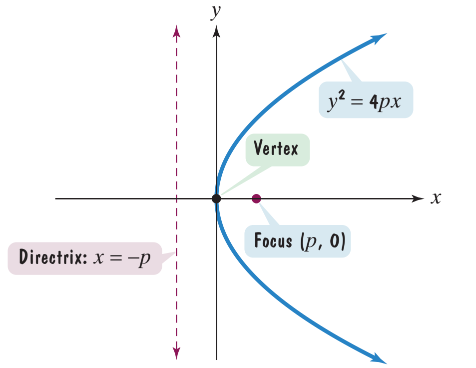


Figure 9.31(a) Parabola with the x-axis as the axis of symmetry. If $p > 0$, the graph opens to the right. If $p < 0$, the graph opens to the left.

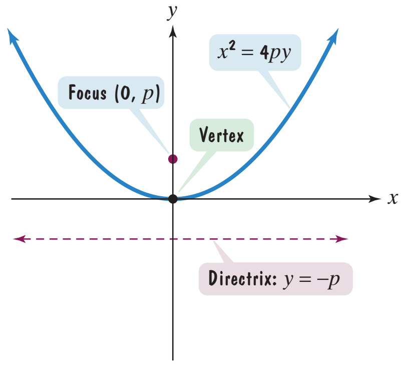


Figure 9.31(b) Parabola with the y-axis as the axis of symmetry. If $p > 0$, the graph opens upward. If $p < 0$, the graph opens downward.

Study Tip

It is helpful to think of p as the *directed distance* from the vertex to the focus. If $p > 0$, the focus lies p units to the right of the vertex or p units above the vertex. If $p < 0$, the focus lies $|p|$ units to the left of the vertex or $|p|$ units below the vertex.

1 Graph parabolas with vertices at the origin.

Using the Standard Form of the Equation of a Parabola

We can use the standard form of the equation of a parabola to find its focus and directrix. Observing the graph's symmetry from its equation is helpful in locating the focus.

$$y^2 = 4px$$

The equation does not change if y is replaced with $-y$. There is x-axis symmetry and the focus is on the x-axis at $(p, 0)$.

$$x^2 = 4py$$

The equation does not change if x is replaced with $-x$. There is y-axis symmetry and the focus is on the y-axis at $(0, p)$.

Although the definition of a parabola is given in terms of its focus and its directrix, the focus and directrix are not part of the graph. The vertex, located at the origin, is a point on the graph of $y^2 = 4px$ and $x^2 = 4py$. Example 1 illustrates how you can find two additional points on the parabola.

EXAMPLE 1 Finding the Focus and Directrix of a Parabola

Find the focus and directrix of the parabola given by $y^2 = 12x$. Then graph the parabola.

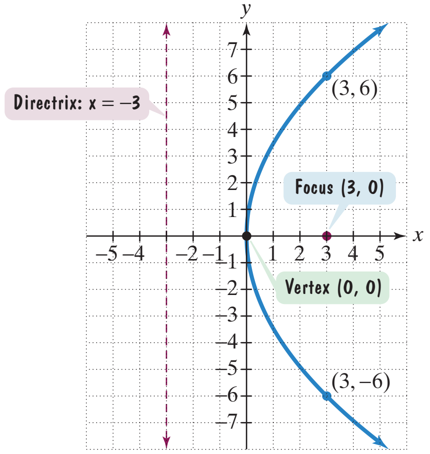
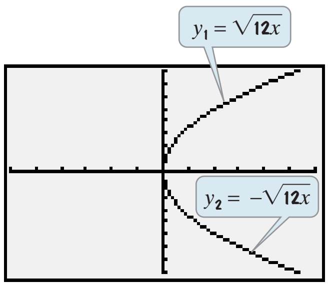


Figure 9.32 The graph of $y^2 = 12x$

Technology

We graph $y^2 = 12x$ with a graphing utility by first solving for y. The screen shows the graphs of $y = \sqrt{12x}$ and $y = -\sqrt{12x}$. The graph fails the vertical line test. Because $y^2 = 12x$ is not a function, you were not familiar with this form of the parabola's equation in Chapter 2.



[−6, 6, 1] by [−8, 8, 1]

Solution The given equation, $y^2 = 12x$, is in the standard form $y^2 = 4px$, so $4p = 12$.

$$y^2 = 12x$$

No change if y is replaced with $-y$. The parabola has x-axis symmetry.

This is $4p$.

We can find both the focus and the directrix by finding p.

$$4p = 12$$

$$p = 3 \qquad \text{Divide both sides by 4.}$$

Because p is positive, the parabola, with its x-axis symmetry, opens to the right. The focus is 3 units to the right of the vertex, $(0, 0)$.

Focus: $(p, 0) = (3, 0)$

Directrix: $x = -p$; $x = -3$

The focus, $(3, 0)$, and directrix, $x = -3$, are shown in **Figure 9.32**.

To graph the parabola, we will use two points on the graph that lie directly above and below the focus. Because the focus is at $(3, 0)$, substitute 3 for x in the parabola's equation, $y^2 = 12x$.

$$y^2 = 12 \cdot 3 \qquad \text{Replace } x \text{ with 3 in } y^2 = 12x.$$

$$y^2 = 36 \qquad \text{Simplify.}$$

$$y = \pm\sqrt{36} = \pm 6 \qquad \text{Apply the square root property.}$$

The points on the parabola above and below the focus are $(3, 6)$ and $(3, -6)$. The graph is sketched in **Figure 9.32**.

Check Point 1 Find the focus and directrix of the parabola given by $y^2 = 8x$. Then graph the parabola.

In general, the points on a parabola $y^2 = 4px$ that lie above and below the focus, $(p, 0)$, are each at a distance $|2p|$ from the focus. This is because if $x = p$, then $y^2 = 4px = 4p^2$, so $y = \pm 2p$. The line segment joining these two points is called the *latus rectum*; its length is $|4p|$.

The Latus Rectum and Graphing Parabolas

The **latus rectum** of a parabola is a line segment that passes through its focus, is parallel to its directrix, and has its endpoints on the parabola. **Figure 9.33** shows that the length of the latus rectum for the graphs of $y^2 = 4px$ and $x^2 = 4py$ is $|4p|$.

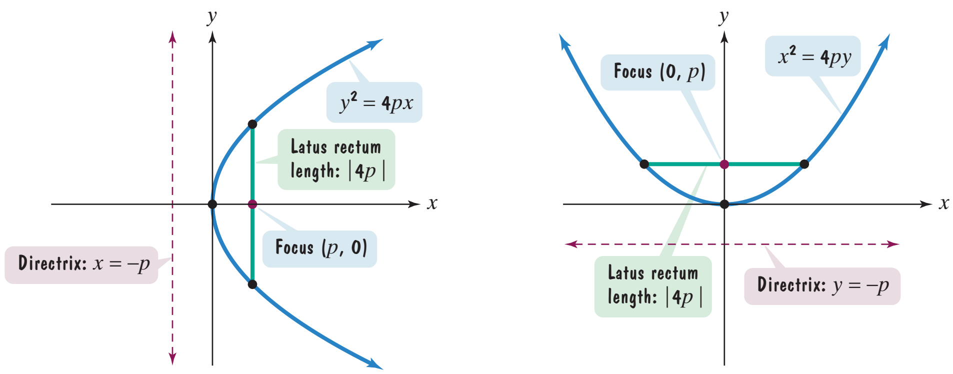


Figure 9.33 Endpoints of the latus rectum are helpful in determining a parabola's "width," or how it opens.

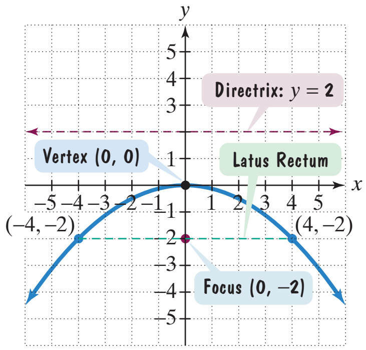
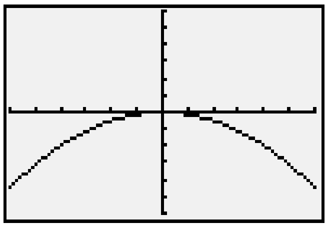


Figure 9.34 The graph of $x^2 = -8y$

Technology

Graph $x^2 = -8y$ by first solving for y: $y = -\frac{x^2}{8}$. The graph passes the vertical line test. Because $x^2 = -8y$ is a function, you were familiar with the parabola's alternate algebraic form, $y = -\frac{1}{8}x^2$, in Chapter 2. The form is $y = ax^2 + bx + c$, with $a = -\frac{1}{8}$, $b = 0$, and $c = 0$.



$[-6, 6, 1]$ by $[-6, 6, 1]$

EXAMPLE 2 Finding the Focus and Directrix of a Parabola

Find the focus and directrix of the parabola given by $x^2 = -8y$. Then graph the parabola.

Solution The given equation, $x^2 = -8y$, is in the standard form $x^2 = 4py$, so $4p = -8$.

$$x^2 = -8y$$

No change if x is replaced with $-x$. The parabola has y-axis symmetry.

This is $4p$.

We can find both the focus and the directrix by finding p.

$$4p = -8$$

$$p = -2 \quad \text{Divide both sides by 4.}$$

Because p is negative, the parabola, with its y-axis symmetry, opens downward. The focus is 2 units below the vertex, $(0, 0)$.

Focus: $(0, p) = (0, -2)$

Directrix: $y = -p$; $y = 2$

The focus and directrix are shown in **Figure 9.34**.

To graph the parabola, we will use the vertex, $(0, 0)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4(-2)| = |-8| = 8.$$

Because the graph has y-axis symmetry, the latus rectum extends 4 units to the left and 4 units to the right of the focus, $(0, -2)$. The endpoints of the latus rectum are $(-4, -2)$ and $(4, -2)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola, shown in **Figure 9.34**.

Check Point 2 Find the focus and directrix of the parabola given by $x^2 = -12y$. Then graph the parabola.

In Examples 1 and 2, we used the equation of a parabola to find its focus and directrix. In the next example, we reverse this procedure.

2 Write equations of parabolas in standard form.

EXAMPLE 3 Finding the Equation of a Parabola from Its Focus and Directrix

Find the standard form of the equation of a parabola with focus $(5, 0)$ and directrix $x = -5$, shown in **Figure 9.35**.

Solution The focus is $(5, 0)$. Thus, the focus is on the x-axis. We use the standard form of the equation in which there is x-axis symmetry, namely $y^2 = 4px$.

We need to determine the value of p. **Figure 9.35** shows that the focus is 5 units to the right of the vertex, $(0, 0)$. Thus, p is positive and $p = 5$. We substitute 5 for p in $y^2 = 4px$ to obtain the standard form of the equation of the parabola. The equation is

$$y^2 = 4 \cdot 5x \quad \text{or} \quad y^2 = 20x.$$

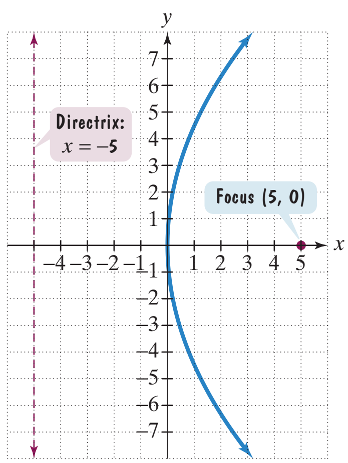


Figure 9.35

Check Point 3 Find the standard form of the equation of a parabola with focus $(8, 0)$ and directrix $x = -8$.

3 Graph parabolas with vertices not at the origin.

Translations of Parabolas

The graph of a parabola can have its vertex at (h, k), rather than at the origin. Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the parabola's equation.

Table 9.3 gives the standard forms of equations of parabolas with vertex at (h, k). **Figure 9.36** shows their graphs.

Table 9.3 Standard Forms of Equations of Parabolas with Vertex at (h, k)

Equation	Vertex	Axis of Symmetry	Focus	Directrix	Description
$(y - k)^2 = 4p(x - h)$	(h, k)	Horizontal	$(h + p, k)$	$x = h - p$	If $p > 0$, opens to the right. If $p < 0$, opens to the left.
$(x - h)^2 = 4p(y - k)$	(h, k)	Vertical	$(h, k + p)$	$y = k - p$	If $p > 0$, opens upward. If $p < 0$, opens downward.

Study Tip

If y is the squared term, there is horizontal symmetry and the parabola's equation is not a function. If x is the squared term, there is vertical symmetry and the parabola's equation is a function. Continue to think of p as the directed distance from the vertex, (h, k), to the focus.

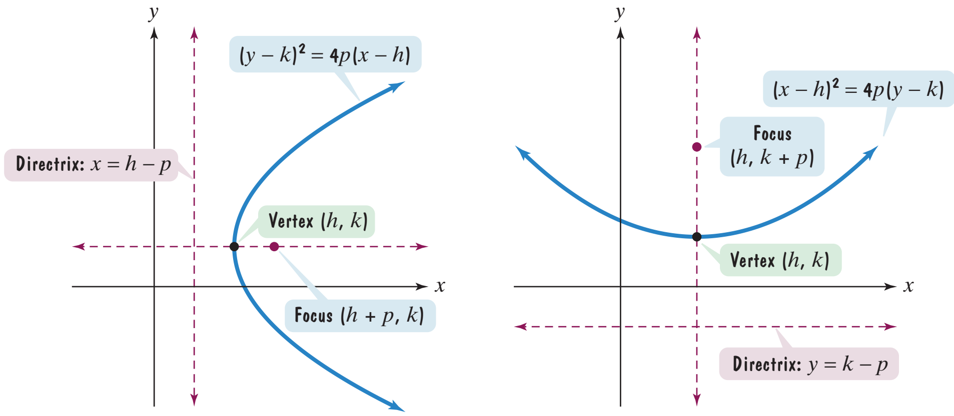


Figure 9.36 Graphs of parabolas with vertex at (h, k) and $p > 0$

The two parabolas shown in **Figure 9.36** illustrate standard forms of equations for $p > 0$. If $p < 0$, a parabola with a horizontal axis of symmetry will open to the left and the focus will lie to the left of the directrix. If $p < 0$, a parabola with a vertical axis of symmetry will open downward and the focus will lie below the directrix.

EXAMPLE 4 Graphing a Parabola with Vertex at (h, k)

Find the vertex, focus, and directrix of the parabola given by

$$(x - 3)^2 = 8(y + 1).$$

Then graph the parabola.

Solution In order to find the focus and directrix, we need to know the vertex. In the standard forms of equations with vertex at (h, k), h is the number subtracted from x and k is the number subtracted from y.

$$(x - 3)^2 = 8(y - (-1))$$

This is $(x - h)^2$, with $h = 3$.

This is $y - k$, with $k = -1$.

We see that $h = 3$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (3, -1)$.

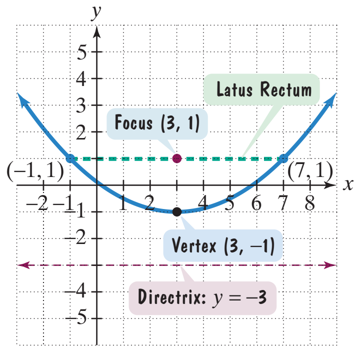


Figure 9.37 The graph of $(x-3)^2 = 8(y+1)$

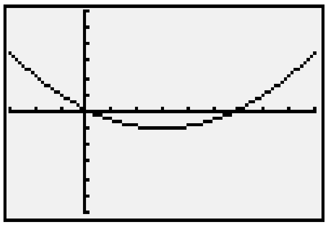
Technology

Graph $(x-3)^2 = 8(y+1)$ by first solving for y:

$$\tfrac{1}{8}(x-3)^2 = y+1$$

$$y = \tfrac{1}{8}(x-3)^2 - 1.$$

The graph passes the vertical line test. Because $(x-3)^2 = 8(y+1)$ is a function, you were familiar with the parabola's alternate algebraic form, $y = \frac{1}{8}(x-3)^2 - 1$, in Chapter 2. The form is $y = a(x-h)^2 + k$ with $a = \frac{1}{8}$, $h = 3$, and $k = -1$.



[−3, 9, 1] by [−6, 6, 1]

Now that we have the vertex, $(3, -1)$, we can find both the focus and directrix by finding p.

$$(x-3)^2 = 8(y+1)$$

The equation is in the standard form $(x-h)^2 = 4p(y-k)$. Because x is the squared term, there is vertical symmetry and the parabola's equation is a function.

This is $4p$.

Because $4p = 8$, $p = 2$. Based on the standard form of the equation, the axis of symmetry is vertical. With a positive value for p and a vertical axis of symmetry, the parabola opens upward. Because $p = 2$, the focus is located 2 units above the vertex, $(3, -1)$. Likewise, the directrix is located 2 units below the vertex.

Focus: $(h, k+p) = (3, -1+2) = (3, 1)$

The vertex, (h, k), is $(3, -1)$.

The focus is 2 units above the vertex, $(3, -1)$.

Directrix: $y = k - p$

$y = -1 - 2 = -3$

The directrix is 2 units below the vertex, $(3, -1)$.

Thus, the focus is $(3, 1)$ and the directrix is $y = -3$. They are shown in **Figure 9.37**. To graph the parabola, we will use the vertex, $(3, -1)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4 \cdot 2| = |8| = 8.$$

Because the graph has vertical symmetry, the latus rectum extends 4 units to the left and 4 units to the right of the focus, $(3, 1)$. The endpoints of the latus rectum are $(3-4, 1)$, or $(-1, 1)$, and $(3+4, 1)$, or $(7, 1)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola, shown in **Figure 9.37**.

Check Point 4 Find the vertex, focus, and directrix of the parabola given by $(x-2)^2 = 4(y+1)$. Then graph the parabola.

In some cases, we need to convert the equation of a parabola to standard form by completing the square on x or y, whichever variable is squared. Let's see how this is done.

EXAMPLE 5 Graphing a Parabola with Vertex at (h, k)

Find the vertex, focus, and directrix of the parabola given by

$$y^2 + 2y + 12x - 23 = 0.$$

Then graph the parabola.

Solution We convert the given equation to standard form by completing the square on the variable y. We isolate the terms involving y on the left side.

$y^2 + 2y + 12x - 23 = 0$ — This is the given equation.

$y^2 + 2y = -12x + 23$ — Isolate the terms involving y.

$y^2 + 2y + 1 = -12x + 23 + 1$ — Complete the square by adding the square of half the coefficient of y.

$(y+1)^2 = -12x + 24$ — Factor.

To express the equation $(y+1)^2 = -12x + 24$ in the standard form $(y-k)^2 = 4p(x-h)$, we factor out -12 on the right. The standard form of the parabola's equation is

$$(y+1)^2 = -12(x-2).$$

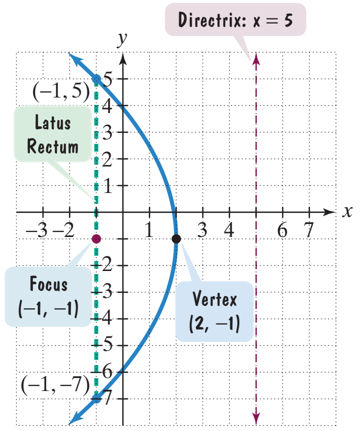


Figure 9.38 The graph of $y^2 + 2y + 12x - 23 = 0$, or $(y + 1)^2 = -12(x - 2)$

Technology

Graph $y^2 + 2y + 12x - 23 = 0$ by solving the equation for y.

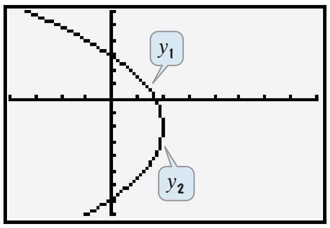
$$y^2 + 2y + (12x - 23) = 0$$

$a = 1$ $b = 2$ $c = 12x - 23$

Use the quadratic formula to solve for y and enter the resulting equations.

$$y_1 = \frac{-2 + \sqrt{4 - 4(12x - 23)}}{2}$$

$$y_2 = \frac{-2 - \sqrt{4 - 4(12x - 23)}}{2}$$



[−4, 8, 1] by [−8, 6, 1]

4 Solve applied problems involving parabolas.

We use $(y + 1)^2 = -12(x - 2)$ to identify the vertex, (h, k), and the value for p needed to locate the focus and the directrix.

$$[y - (-1)]^2 = -12(x - 2)$$

This is $(y - k)^2$, with $k = -1$. This is $4p$. This is $x - h$, with $h = 2$.

The equation is in the standard form $(y - k)^2 = 4p(x - h)$. Because y is the squared term, there is horizontal symmetry and the parabola's equation is not a function.

We see that $h = 2$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (2, -1)$. Because $4p = -12$, $p = -3$. Based on the standard form of the equation, the axis of symmetry is horizontal. With a negative value for p and a horizontal axis of symmetry, the parabola opens to the left. Because $p = -3$, the focus is located 3 units to the left of the vertex, $(2, -1)$. Likewise, the directrix is located 3 units to the right of the vertex.

Focus: $(h + p, k) = (2 + (-3), -1) = (-1, -1)$

The vertex, (h, k), is $(2, -1)$. The focus is 3 units to the left of the vertex, $(2, -1)$.

Directrix: $x = h - p$

$x = 2 - (-3) = 5$

The directrix is 3 units to the right of the vertex, $(2, -1)$.

Thus, the focus is $(-1, -1)$ and the directrix is $x = 5$. They are shown in **Figure 9.38**.

To graph the parabola, we will use the vertex, $(2, -1)$, and the two endpoints of the latus rectum. The length of the latus rectum is

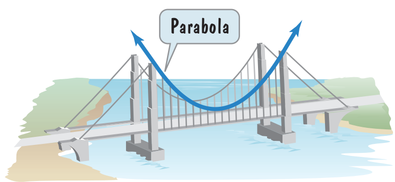
$$|4p| = |4(-3)| = |-12| = 12.$$

Because the graph has horizontal symmetry, the latus rectum extends 6 units above and 6 units below the focus, $(-1, -1)$. The endpoints of the latus rectum are $(-1, -1 + 6)$, or $(-1, 5)$, and $(-1, -1 - 6)$, or $(-1, -7)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola shown in **Figure 9.38**.

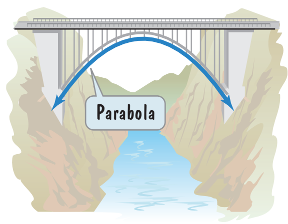
Check Point 5 Find the vertex, focus, and directrix of the parabola given by $y^2 + 2y + 4x - 7 = 0$. Then graph the parabola.

Applications

Parabolas have many applications. Cables hung between structures to form suspension bridges form parabolas. Arches constructed of steel and concrete, whose main purpose is strength, are usually parabolic in shape.



Suspension bridge



Arch bridge

We have seen that comets in our solar system travel in orbits that are ellipses and hyperbolas. Some comets follow parabolic paths. Only comets with elliptical orbits, such as Halley's Comet, return to our part of the galaxy.

If a parabola is rotated about its axis of symmetry, a parabolic surface is formed. **Figure 9.39(a)** shows how a parabolic surface can be used to reflect light. Light originates at the focus. Note how the light is reflected by the parabolic surface, so that the outgoing light is parallel to the axis of symmetry. The reflective properties of parabolic surfaces are used in the design of searchlights [see **Figure 9.39(b)**], automobile headlights, and parabolic microphones.

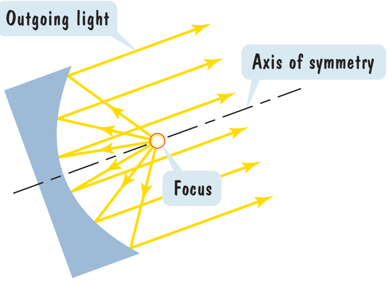


Figure 9.39(a) Parabolic surface reflecting light

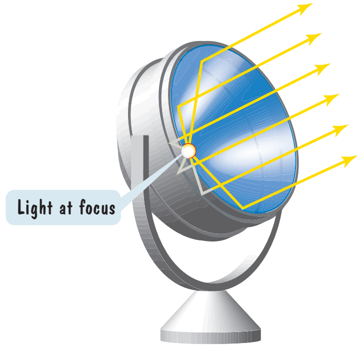


Figure 9.39(b) Light from the focus is reflected parallel to the axis of symmetry.

Figure 9.40(a) shows how a parabolic surface can be used to reflect *incoming* light. Note that light rays strike the surface and are reflected *to the focus*. This principle is used in the design of reflecting telescopes, radar, and television satellite dishes. Reflecting telescopes magnify the light from distant stars by reflecting the light from these bodies to the focus of a parabolic mirror [see **Figure 9.40(b)**].

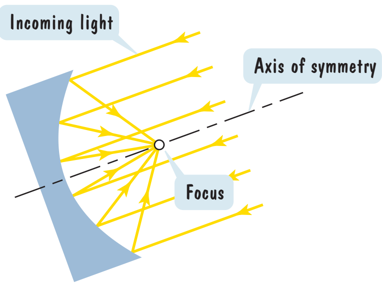


Figure 9.40(a) Parabolic surface reflecting incoming light

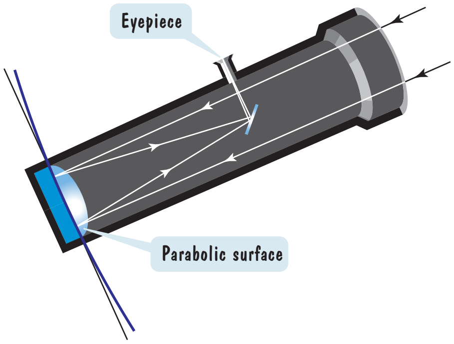


Figure 9.40(b) Incoming light rays are reflected to the focus.

The Hubble Space Telescope

The Hubble Space Telescope

For decades, astronomers hoped to create an observatory above the atmosphere that would provide an unobscured view of the universe. This vision was realized with the 1990 launching of the Hubble Space Telescope. The telescope initially had blurred vision due to problems with its parabolic mirror. The mirror had been ground two millionths of a meter smaller than design specifications. In 1993, astronauts from the Space Shuttle *Endeavor* equipped the telescope with optics to correct the blurred vision. "A small change for a mirror, a giant leap for astronomy," Christopher J. Burrows of the Space Telescope Science Institute said when clear images from the ends of the universe were presented to the public after the repair mission.

EXAMPLE 6 Using the Reflection Property of Parabolas

An engineer is designing a flashlight using a parabolic reflecting mirror and a light source, shown in **Figure 9.41**. The casting has a diameter of 4 inches and a depth of 2 inches. What is the equation of the parabola used to shape the mirror? At what point should the light source be placed relative to the mirror's vertex?

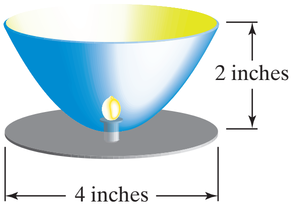


Figure 9.41 Designing a flashlight

Solution We position the parabola with its vertex at the origin and opening upward (**Figure 9.42**). Thus, the focus is on the y-axis, located at $(0, p)$. We use the

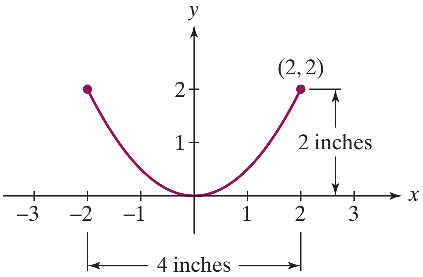


Figure 9.42

standard form of the equation in which there is y-axis symmetry, namely $x^2 = 4py$. We need to find p. Because (2, 2) lies on the parabola, we let $x = 2$ and $y = 2$ in $x^2 = 4py$.

$$2^2 = 4p \cdot 2 \quad \text{Substitute 2 for } x \text{ and 2 for } y \text{ in } x^2 = 4py.$$

$$4 = 8p \quad \text{Simplify.}$$

$$p = \tfrac{1}{2} \quad \text{Divide both sides of the equation by 8 and reduce the resulting fraction.}$$

We substitute $\frac{1}{2}$ for p in $x^2 = 4py$ to obtain the standard form of the equation of the parabola. The equation of the parabola used to shape the mirror is

$$x^2 = 4 \cdot \tfrac{1}{2} y \qquad \text{or} \qquad x^2 = 2y.$$

The light source should be placed at the focus, $(0, p)$. Because $p = \frac{1}{2}$, the light should be placed at $\left(0, \frac{1}{2}\right)$, or $\frac{1}{2}$ inch above the vertex.

Check Point 6 In Example 6, suppose that the casting has a diameter of 6 inches and a depth of 4 inches. What is the equation of the parabola used to shape the mirror? At what point should the light source be placed relative to the mirror's vertex?

Degenerate Conic Sections

We opened the chapter by noting that conic sections are curves that result from the intersection of a cone and a plane. However, these intersections might not result in a conic section. Three degenerate cases occur when the cutting plane passes through the vertex. These **degenerate conic sections** are a point, a line, and a pair of intersecting lines, illustrated in **Figure 9.43**.

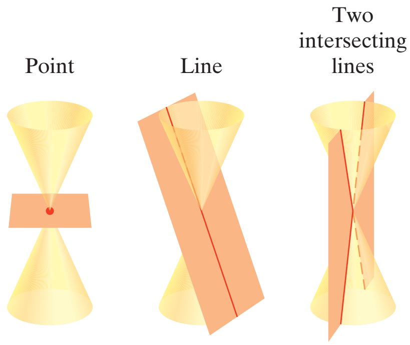


Figure 9.43 Degenerate conics

Exercise Set 9.3

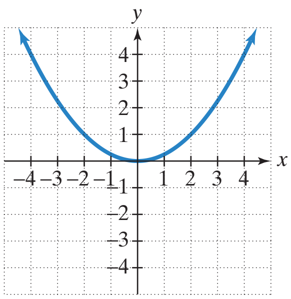
Practice Exercises

In Exercises 1–4, find the focus and directrix of each parabola with the given equation. Then match each equation to one of the graphs that are shown and labeled (a)–(d).

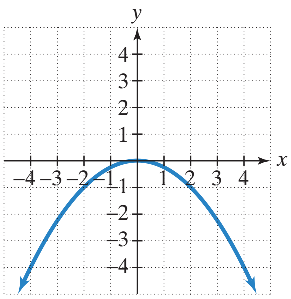
1. $y^2 = 4x$ **2.** $x^2 = 4y$

3. $x^2 = -4y$ **4.** $y^2 = -4x$

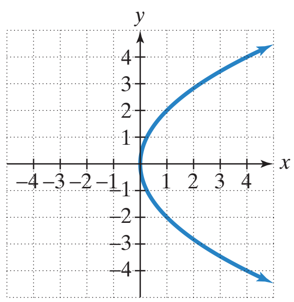
a.



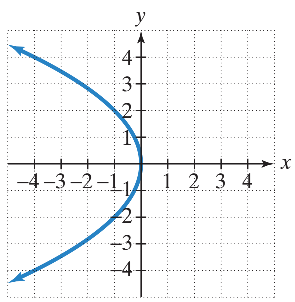
b.



c.



d.



In Exercises 5–16, find the focus and directrix of the parabola with the given equation. Then graph the parabola.

5. $y^2 = 16x$ **6.** $y^2 = 4x$

7. $y^2 = -8x$ **8.** $y^2 = -12x$

9. $x^2 = 12y$ **10.** $x^2 = 8y$

11. $x^2 = -16y$ **12.** $x^2 = -20y$

13. $y^2 - 6x = 0$ **14.** $x^2 - 6y = 0$

15. $8x^2 + 4y = 0$ **16.** $8y^2 + 4x = 0$

In Exercises 17–30, find the standard form of the equation of each parabola satisfying the given conditions.

17. Focus: $(7, 0)$; Directrix: $x = -7$

18. Focus: $(9, 0)$; Directrix: $x = -9$

19. Focus: $(-5, 0)$; Directrix: $x = 5$

20. Focus: $(-10, 0)$; Directrix: $x = 10$

21. Focus: $(0, 15)$; Directrix: $y = -15$

22. Focus: $(0, 20)$; Directrix: $y = -20$

23. Focus: $(0, -25)$; Directrix: $y = 25$

24. Focus: $(0, -15)$; Directrix: $y = 15$

25. Vertex: $(2, -3)$; Focus: $(2, -5)$

26. Vertex: $(5, -2)$; Focus: $(7, -2)$

27. Focus: $(3, 2)$; Directrix: $x = -1$

28. Focus: $(2, 4)$; Directrix: $x = -4$

29. Focus: $(-3, 4)$; Directrix: $y = 2$

30. Focus: $(7, -1)$; Directrix: $y = -9$

In Exercises 31–34, find the vertex, focus, and directrix of each parabola with the given equation. Then match each equation to one of the graphs that are shown and labeled (a)–(d).

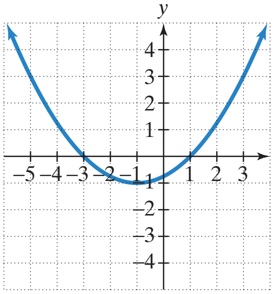
31. $(y - 1)^2 = 4(x - 1)$

32. $(x + 1)^2 = 4(y + 1)$

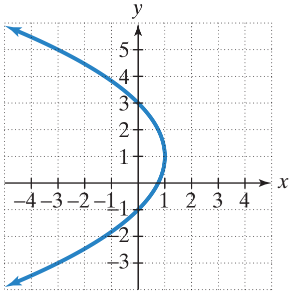
33. $(x + 1)^2 = -4(y + 1)$

34. $(y - 1)^2 = -4(x - 1)$

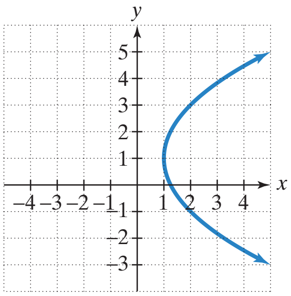
a.



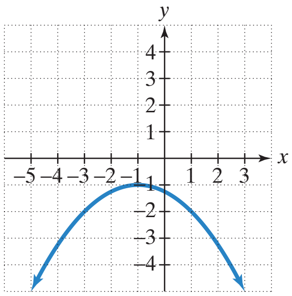
b.



c.



d.



In Exercises 35–42, find the vertex, focus, and directrix of each parabola with the given equation. Then graph the parabola.

35. $(x - 2)^2 = 8(y - 1)$

36. $(x + 2)^2 = 4(y + 1)$

37. $(x + 1)^2 = -8(y + 1)$

38. $(x + 2)^2 = -8(y + 2)$

39. $(y + 3)^2 = 12(x + 1)$

40. $(y + 4)^2 = 12(x + 2)$

41. $(y + 1)^2 = -8x$

42. $(y - 1)^2 = -8x$

In Exercises 43–48, convert each equation to standard form by completing the square on x or y. Then find the vertex, focus, and directrix of the parabola. Finally, graph the parabola.

43. $x^2 - 2x - 4y + 9 = 0$

44. $x^2 + 6x + 8y + 1 = 0$

45. $y^2 - 2y + 12x - 35 = 0$

46. $y^2 - 2y - 8x + 1 = 0$

47. $x^2 + 6x - 4y + 1 = 0$

48. $x^2 + 8x - 4y + 8 = 0$

Practice Plus

In Exercises 49–54, use the vertex and the direction in which the parabola opens to determine the relation's domain and range. Is the relation a function?

49. $y^2 + 6y - x + 5 = 0$

50. $y^2 - 2y - x - 5 = 0$

51. $y = -x^2 + 4x - 3$

52. $y = -x^2 - 4x + 4$

53. $x = -4(y - 1)^2 + 3$

54. $x = -3(y - 1)^2 - 2$

In Exercises 55–60, find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

55. $\begin{cases} (y - 2)^2 = x + 4 \\ y = -\frac{1}{2}x \end{cases}$

56. $\begin{cases} (y - 3)^2 = x - 2 \\ x + y = 5 \end{cases}$

57. $\begin{cases} x = y^2 - 3 \\ x = y^2 - 3y \end{cases}$

58. $\begin{cases} x = y^2 - 5 \\ x^2 + y^2 = 25 \end{cases}$

59. $\begin{cases} x = (y + 2)^2 - 1 \\ (x - 2)^2 + (y + 2)^2 = 1 \end{cases}$

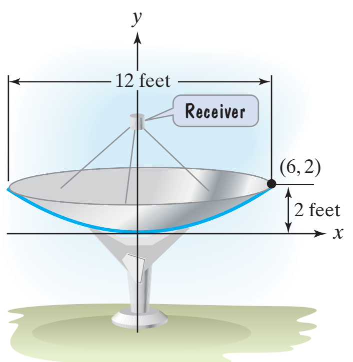
60. $\begin{cases} x = 2y^2 + 4y + 5 \\ (x + 1)^2 + (y - 2)^2 = 1 \end{cases}$

Application Exercises

61. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 4 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?

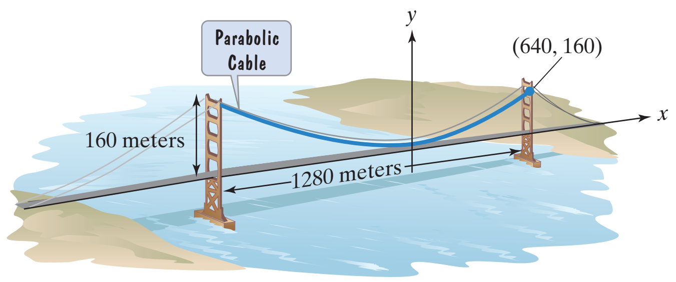
62. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 8 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?

63. A satellite dish, like the one shown below, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver is located. The satellite dish shown has a diameter of 12 feet and a depth of 2 feet. How far from the base of the dish should the receiver be placed?

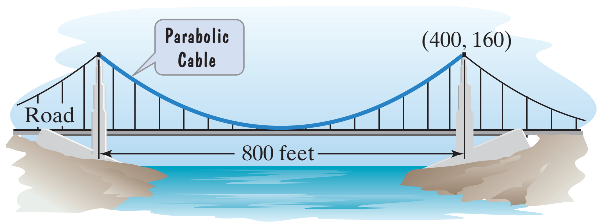


64. In Exercise 63, if the diameter of the dish is halved and the depth stays the same, how far from the base of the smaller dish should the receiver be placed?

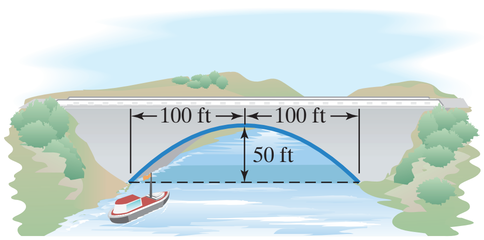
65. The towers of the Golden Gate Bridge connecting San Francisco to Marin County are 1280 meters apart and rise 160 meters above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 200 meters from a tower? Round to the nearest meter.



66. The towers of a suspension bridge are 800 feet apart and rise 160 feet above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 feet from a tower?



67. The parabolic arch shown in the figure is 50 feet above the water at the center and 200 feet wide at the base. Will a boat that is 30 feet tall clear the arch 30 feet from the center?



68. A satellite dish in the shape of a parabolic surface has a diameter of 20 feet. If the receiver is to be placed 6 feet from the base, how deep should the dish be?

Writing in Mathematics

69. What is a parabola?

70. Explain how to use $y^2 = 8x$ to find the parabola's focus and directrix.

71. If you are given the standard form of the equation of a parabola with vertex at the origin, explain how to determine if the parabola opens to the right, left, upward, or downward.

72. Describe one similarity and one difference between the graphs of $y^2 = 4x$ and $(y - 1)^2 = 4(x - 1)$.

73. How can you distinguish parabolas from other conic sections by looking at their equations?

74. Look at the satellite dish shown in Exercise 63. Why must the receiver for a shallow dish be farther from the base of the dish than for a deeper dish of the same diameter?

Technology Exercises

75. Use a graphing utility to graph any five of the parabolas that you graphed by hand in Exercises 5–16.

76. Use a graphing utility to graph any three of the parabolas that you graphed by hand in Exercises 35–42. First solve the given equation for y, possibly using the square root property. Enter each of the two resulting equations to produce the complete graph.

Use a graphing utility to graph the parabolas in Exercises 77–78. Write the given equation as a quadratic equation in y and use the quadratic formula to solve for y. Enter each of the equations to produce the complete graph.

77. $y^2 + 2y - 6x + 13 = 0$ **78.** $y^2 + 10y - x + 25 = 0$

In Exercises 79–80, write each equation as a quadratic equation in y and then use the quadratic formula to express y in terms of x. Graph the resulting two equations using a graphing utility. What effect does the xy-term have on the graph of the resulting parabola?

79. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

80. $x^2 + 2\sqrt{3}xy + 3y^2 + 8\sqrt{3}x - 8y + 32 = 0$

Critical Thinking Exercises

Make Sense? *In Exercises 81–84, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

81. I graphed a parabola that opened to the right and contained a maximum point.

82. Knowing that a parabola opening to the right has a vertex at $(-1, 1)$ gives me enough information to determine its graph.

83. I noticed that depending on the values for A and B, assuming that they are both not zero, the graph of $Ax^2 + By^2 = C$ can represent any of the conic sections other than a parabola.

84. I'm using a telescope in which light from distant stars is reflected to the focus of a parabolic mirror.

In Exercises 85–88, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

85. The parabola whose equation is $x = 2y - y^2 + 5$ opens to the right.

86. If the parabola whose equation is $x = ay^2 + by + c$ has its vertex at $(3, 2)$ and $a > 0$, then it has no y-intercepts.

87. Some parabolas that open to the right have equations that define y as a function of x.

88. The graph of $x = a(y - k) + h$ is a parabola with vertex at (h, k).

89. Find the focus and directrix of a parabola whose equation is of the form $Ax^2 + Ey = 0$, $A \neq 0$, $E \neq 0$.

90. Write the standard form of the equation of a parabola whose points are equidistant from $y = 4$ and $(-1, 0)$.

Group Exercise

91. Consult the research department of your library or the Internet to find an example of architecture that incorporates one or more conic sections in its design. Share this example with other group members. Explain precisely how conic sections are used. Do conic sections enhance the appeal of the architecture? In what ways?

Preview Exercises

Exercises 92–94 will help you prepare for the material covered in the next section.

92. Simplify and write the equation in standard form in terms of x' and y':

$$\left[\frac{\sqrt{2}}{2}(x' - y')\right]\left[\frac{\sqrt{2}}{2}(x' + y')\right] = 1.$$

93. a. Make a sketch showing that $\cot 2\theta = -\frac{7}{24}$ for $90° < 2\theta < 180°$.

b. Use your sketch from part (a) to determine the value of $\cos 2\theta$.

c. Use the value of $\cos 2\theta$ from part (b) and the identities

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \quad \text{and} \quad \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$$

to determine the values of $\sin\theta$ and $\cos\theta$.

d. In part (c), why did we not write $\pm$ before the radical in each formula?

94. The equation $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$ is in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Use the equation to determine the value of $B^2 - 4AC$.

Chapter 9 Mid-Chapter Check Point

What You Know: We learned that the four conic sections are the circle, the ellipse, the hyperbola, and the parabola. Prior to this chapter, we graphed circles with center (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2.$$

In this chapter, you learned to graph ellipses centered at the origin and ellipses centered at (h, k):

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,\ a^2 > b^2.$$

We saw that the larger denominator (a^2) determines whether the major axis is horizontal or vertical. We used vertices and asymptotes to graph hyperbolas centered at the origin and hyperbolas centered at (h, k):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{or} \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

We used $c^2 = a^2 - b^2$ to locate the foci of an ellipse. We used $c^2 = a^2 + b^2$ to locate the foci of a hyperbola. Finally, we used the vertex and the latus rectum to graph parabolas with vertices at the origin and parabolas with vertices at (h, k):

$$(y-k)^2 = 4p(x-h) \quad \text{or} \quad (x-h)^2 = 4p(y-k).$$

In Exercises 1–5, graph each ellipse. Give the location of the foci.

1. $\frac{x^2}{25} + \frac{y^2}{4} = 1$

2. $9x^2 + 4y^2 = 36$

3. $\frac{(x-2)^2}{16} + \frac{(y+1)^2}{25} = 1$

4. $\frac{(x+2)^2}{25} + \frac{(y-1)^2}{16} = 1$

5. $x^2 + 9y^2 - 4x + 54y + 49 = 0$

In Exercises 6–11, graph each hyperbola. Give the location of the foci and the equations of the asymptotes.

6. $\frac{x^2}{9} - y^2 = 1$

7. $\frac{y^2}{9} - x^2 = 1$

8. $y^2 - 4x^2 = 16$

9. $4x^2 - 49y^2 = 196$

10. $\frac{(x-2)^2}{9} - \frac{(y+2)^2}{16} = 1$

11. $4x^2 - y^2 + 8x + 6y + 11 = 0$

In Exercises 12–13, graph each parabola. Give the location of the focus and the directrix.

12. $(x-2)^2 = -12(y+1)$

13. $y^2 - 2x - 2y - 5 = 0$

In Exercises 14–21, graph each equation.

14. $x^2 + y^2 = 4$

15. $x + y = 4$

16. $x^2 - y^2 = 4$

17. $x^2 + 4y^2 = 4$

18. $(x+1)^2 + (y-1)^2 = 4$

19. $x^2 + 4(y-1)^2 = 4$

20. $(x-1)^2 - (y-1)^2 = 4$

21. $(y+1)^2 = 4(x-1)$

In Exercises 22–27, find the standard form of the equation of the conic section satisfying the given conditions.

22. Ellipse; Foci: $(-4, 0)$, $(4, 0)$; Vertices: $(-5, 0)$, $(5, 0)$

23. Ellipse; Endpoints of major axis: $(-8, 2)$, $(10, 2)$; Foci: $(-4, 2)$, $(6, 2)$

24. Hyperbola; Foci: $(0, -3)$, $(0, 3)$; Vertices: $(0, -2)$, $(0, 2)$

25. Hyperbola; Foci: $(-4, 5)$, $(2, 5)$; Vertices: $(-3, 5)$, $(1, 5)$

26. Parabola; Focus: $(4, 5)$; Directrix: $y = -1$

27. Parabola; Focus: $(-2, 6)$; Directrix: $x = 8$

28. A semielliptical archway over a one-way road has a height of 10 feet and a width of 30 feet. A truck has a width of 10 feet and a height of 9.5 feet. Will this truck clear the opening of the archway?

29. A lithotriper is used to disentegrate kidney stones. The patient is placed within an elliptical device with the kidney centered at one focus, while ultrasound waves from the other focus hit the walls and are reflected to the kidney stone, shattering the stone. Suppose that the length of the major axis of the ellipse is 40 centimeters and the length of the minor axis is 20 centimeters. How far from the kidney stone should the electrode that sends the ultrasound waves be placed in order to shatter the stone?

30. An explosion is recorded by two forest rangers, one at a primary station and the other at an outpost 6 kilometers away. The ranger at the primary station hears the explosion 6 seconds before the ranger at the outpost.

a. Assuming sound travels at 0.35 kilometer per second, write an equation in standard form that gives all the possible locations of the explosion. Use a coordinate system with the two ranger stations on the x-axis and the midpoint between the stations at the origin.

b. Graph the equation that gives the possible locations of the explosion. Show the locations of the ranger stations in your drawing.

31. A domed ceiling is a parabolic surface. Ten meters down from the top of the dome, the ceiling is 15 meters wide. For the best lighting on the floor, a light source should be placed at the focus of the parabolic surface. How far from the top of the dome, to the nearest tenth of a meter, should the light source be placed?