Section 9.4 Rotation of Axes

Objectives

1. Identify conics without completing the square.
2. Use rotation of axes formulas.
3. Write equations of rotated conics in standard form.
4. Identify conics without rotating axes.

Richard E. Prince "The Cone of Apollonius" (detail), fiberglass, steel, paint, graphite, 51 × 18 × 14 in. Collection: Vancouver Art Gallery, Vancouver, Canada. Photo courtesy of Equinox Gallery, Vancouver, Canada.

To recognize a conic section, you often need to pay close attention to its graph. Graphs powerfully enhance our understanding of algebra and trigonometry. However, it is not possible for people who are blind—or sometimes, visually impaired—to see a graph. Creating informative materials for the blind and visually impaired is a challenge for instructors and mathematicians. Many people who are visually impaired "see" a graph by touching a three-dimensional representation of that graph, perhaps while it is described verbally.

Is it possible to identify conic sections in nonvisual ways? The answer is yes, and the methods for doing so are related to the coefficients in their equations. As we present these methods, think about how you learn them. How would your approach to studying mathematics change if we removed all graphs and replaced them with verbal descriptions?

1 Identify conics without completing the square.

Identifying Conic Sections without Completing the Square

Conic sections can be represented both geometrically (as intersecting planes and cones) and algebraically. The equations of the conic sections we have considered in the first three sections of this chapter can be expressed in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

in which A and C are not both zero. You can use A and C, the coefficients of x^2 and y^2, respectively, to identify a conic section without completing the square.

Identifying a Conic Section without Completing the Square

A nondegenerate conic section of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

in which A and C are not both zero, is

- a circle if $A = C$,
- a parabola if $AC = 0$,
- an ellipse if $A \neq C$ and $AC > 0$, and
- a hyperbola if $AC < 0$.

EXAMPLE 1 Identifying a Conic Section without Completing the Square

Identify the graph of each of the following nondegenerate conic sections:

a. $4x^2 - 25y^2 - 24x + 250y - 489 = 0$

b. $x^2 + y^2 + 6x - 2y + 6 = 0$

c. $y^2 + 12x + 2y - 23 = 0$

d. $9x^2 + 25y^2 - 54x + 50y - 119 = 0.$

Solution We use A, the coefficient of x^2, and C, the coefficient of y^2, to identify each conic section.

a. $4x^2 - 25y^2 - 24x + 250y - 489 = 0$

$A = 4$ $C = -25$

$AC = 4(-25) = -100 < 0$

Because $AC < 0$, the graph of the equation is a hyperbola.

b. $x^2 + y^2 + 6x - 2y + 6 = 0$

$A = 1$ $C = 1$

Because $A = C$, the graph of the equation is a circle.

c. We can write $y^2 + 12x + 2y - 23 = 0$ as

$$0x^2 + y^2 + 12x + 2y - 23 = 0.$$

$A = 0$ $C = 1$

$AC = 0(1) = 0$

Because $AC = 0$, the graph of the equation is a parabola.

d. $9x^2 + 25y^2 - 54x + 50y - 119 = 0$

$A = 9$ $C = 25$

$AC = 9(25) = 225 > 0.$

Because $AC > 0$ and $A \neq C$, the graph of the equation is an ellipse.

Check Point 1 Identify the graph of each of the following nondegenerate conic sections:

a. $3x^2 + 2y^2 + 12x - 4y + 2 = 0$
b. $x^2 + y^2 - 6x + y + 3 = 0$
c. $y^2 - 12x - 4y + 52 = 0$
d. $9x^2 - 16y^2 - 90x + 64y + 17 = 0.$

2 Use rotation of axes formulas.

Rotation of Axes

Figure 9.44 shows the graph of

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0.$$

The graph looks like an ellipse, although its major axis neither lies along the x-axis or y-axis nor is parallel to the x-axis or y-axis. Do you notice anything unusual about the equation? It contains an xy-term. However, look at what happens if we rotate the x- and y-axes through an angle of 30°. In the rotated $x'y'$-system, the major axis of the ellipse lies along the x'-axis. We can write the equation of the ellipse in this rotated $x'y'$-system as

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1.$$

Observe that there is no $x'y'$-term in the equation.

Except for degenerate cases, the **general second-degree equation**

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

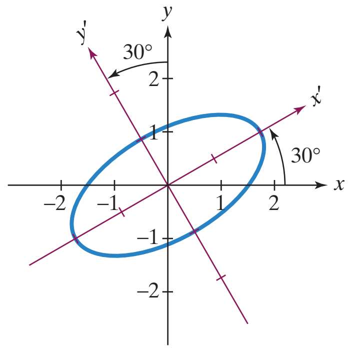
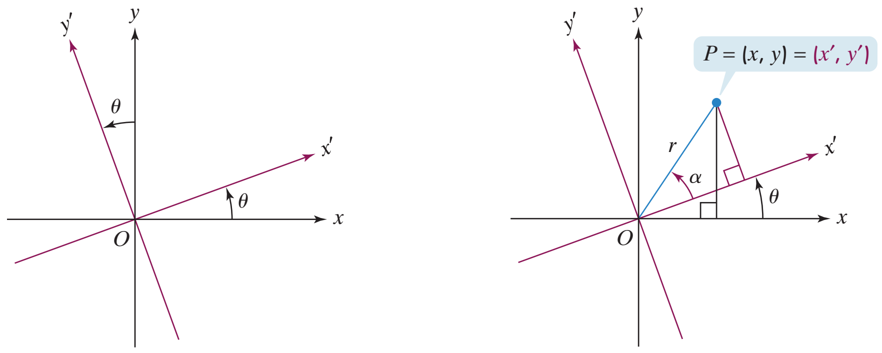


Figure 9.44 The graph of $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$, a rotated ellipse

represents one of the conic sections. However, due to the xy-term in the equation, these conic sections are rotated in such a way that their axes are no longer parallel to the x- and y-axes. To reduce these equations to forms of the conic sections with which you are already familiar, we use a procedure called **rotation of axes**.

Suppose that the x- and y-axes are rotated through a positive angle θ, resulting in a new $x'y'$ coordinate system. This system is shown in **Figure 9.45(a)**. The origin in the $x'y'$-system is the same as the origin in the xy-system. Point P in **Figure 9.45(b)** has coordinates (x, y) relative to the xy-system and coordinates (x', y') relative to the $x'y'$-system. Our goal is to obtain formulas relating the old and new coordinates. Thus, we need to express x and y in terms of x', y', and θ.



(a) Rotating the x- and y-axes through a positive angle θ

(b) Describing point P relative to the xy-system and the rotated $x'y'$-system

Figure 9.45 Rotating axes

Look at **Figure 9.45(b)**. Notice that

$r =$ the distance from the origin O to point P.

$\alpha =$ the angle from the positive x'-axis to the ray from O through P.

Using the definitions of sine and cosine, we obtain

$$\cos\alpha = \frac{x'}{r}: x' = r\cos\alpha$$

$$\sin\alpha = \frac{y'}{r}: y' = r\sin\alpha$$

This is from the right triangle with a leg along the x'-axis.

$$\cos(\theta + \alpha) = \frac{x}{r}: x = r\cos(\theta + \alpha)$$

$$\sin(\theta + \alpha) = \frac{y}{r}: y = r\sin(\theta + \alpha).$$

This is from the taller right triangle with a leg along the x-axis.

Thus,

$$x = r\cos(\theta + \alpha)$$

This is the third of the preceding equations.

$$= r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

Use the formula for the cosine of the sum of two angles.

$$= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$$

Apply the distributive property and rearrange factors.

$$= x'\cos\theta - y'\sin\theta.$$

Use the first and second of the preceding equations: $x' = r\cos\alpha$ and $y' = r\sin\alpha$.

Similarly,

$$y = r\sin(\theta + \alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = x'\sin\theta + y'\cos\theta.$$

Rotation of Axes Formulas

Suppose an xy-coordinate system and an $x'y'$-coordinate system have the same origin and θ is the angle from the positive x-axis to the positive x'-axis. If the coordinates of point P are (x, y) in the xy-system and (x', y') in the rotated $x'y'$-system, then

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta.$$

EXAMPLE 2 Rotating Axes

Write the equation $xy = 1$ in terms of a rotated $x'y'$-system if the angle of rotation from the x-axis to the x'-axis is 45°. Express the equation in standard form. Use the rotated system to graph $xy = 1$.

Solution With $\theta = 45°$, the rotation formulas for x and y are

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta = x' \cos 45° - y' \sin 45° \\ &= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y') \\ y &= x' \sin\theta + y' \cos\theta = x' \sin 45° + y' \cos 45° \\ &= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y'). \end{aligned}$$

Now substitute these expressions for x and y in the given equation, $xy = 1$.

$$xy = 1 \qquad \text{This is the given equation.}$$

$$\left[\frac{\sqrt{2}}{2}(x' - y')\right]\left[\frac{\sqrt{2}}{2}(x' + y')\right] = 1 \qquad \text{Substitute the expressions for } x \text{ and } y \text{ from the rotation formulas.}$$

$$\frac{2}{4}(x' - y')(x' + y') = 1 \qquad \text{Multiply: } \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{2}{4}.$$

$$\frac{1}{2}(x'^2 - y'^2) = 1 \qquad \text{Reduce } \frac{2}{4} \text{ and multiply the binomials.}$$

$$\underbrace{\frac{x'^2}{2}}_{a^2 = 2} - \underbrace{\frac{y'^2}{2}}_{b^2 = 2} = 1 \qquad \text{Write the equation in standard form: } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

This equation expresses $xy = 1$ in terms of the rotated $x'y'$-system. Can you see that this is the standard form of the equation of a hyperbola? The hyperbola's center is at (0, 0), with the transverse axis on the x'-axis. The vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 2$, the vertices are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$, located on the x'-axis. Based on the standard form of the hyperbola's equation, the equations for the asymptotes are

$$y' = \pm\frac{b}{a}x' \quad \text{or} \quad y' = \pm\frac{\sqrt{2}}{\sqrt{2}}x'.$$

The equations of the asymptotes can be simplified to $y' = x'$ and $y' = -x'$, which correspond to the original x- and y-axes. The graph of the hyperbola is shown in **Figure 9.46**.

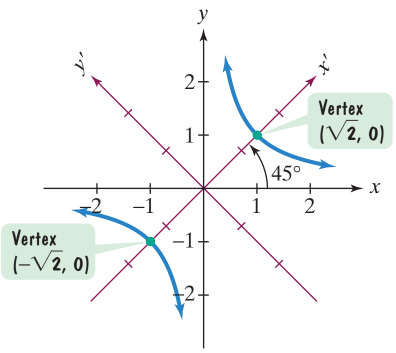


Figure 9.46 The graph of $xy = 1$ or $\frac{x'^2}{2} - \frac{y'^2}{2} = 1$

Check Point 2 Write the equation $xy = 2$ in terms of a rotated $x'y'$-system if the angle of rotation from the x-axis to the x'-axis is 45°. Express the equation in standard form. Use the rotated system to graph $xy = 2$.

3 Write equations of rotated conics in standard form.

Using Rotations to Transform Equations with *xy*-Terms to Standard Equations of Conic Sections

We have noted that the appearance of the term Bxy $(B \neq 0)$ in the general second-degree equation indicates that the graph of the conic section has been rotated. A rotation of axes through an appropriate angle can transform the equation to one of the standard forms of the conic sections in x' and y' in which no $x'y'$-term appears.

Amount of Rotation Formula

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0$$

can be rewritten as an equation in x' and y' without an $x'y'$-term by rotating the axes through angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

Before we learn to apply this formula, let's see how it can be derived. We begin with the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0.$$

Then we rotate the axes through an angle θ. In terms of the rotated $x'y'$-system, the general second-degree equation can be written as

$$\begin{aligned} A(x' \cos\theta - y' \sin\theta)^2 &+ B(x' \cos\theta - y' \sin\theta)(x' \sin\theta + y' \cos\theta) \\ &+ C(x' \sin\theta + y' \cos\theta)^2 + D(x' \cos\theta - y' \sin\theta) \\ &+ E(x' \sin\theta + y' \cos\theta) + F = 0. \end{aligned}$$

After a lot of simplifying that involves expanding and collecting like terms, you will obtain the following equation:

We want a rotation that results in no $x'y'$-term.

$$\begin{aligned} (A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta)x'^2 &+ [B(\cos^2\theta - \sin^2\theta) + 2(C - A)(\sin\theta\cos\theta)]x'y' \\ &+ (A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta)y'^2 \\ &+ (D\cos\theta + E\sin\theta)x' \\ &+ (-D\sin\theta + E\cos\theta)y' + F = 0. \end{aligned}$$

If this looks somewhat ghastly, take a deep breath and focus only on the $x'y'$-term. We want to choose θ so that the coefficient of this term is zero. This will give the required rotation that results in no $x'y'$-term.

$$B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta = 0$$

Set the coefficient of the $x'y'$-term equal to 0.

$$B\cos 2\theta + (C - A)\sin 2\theta = 0$$

Use the double-angle formulas: $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

$$B\cos 2\theta = -(C - A)\sin 2\theta$$

Subtract $(C - A)\sin 2\theta$ from both sides.

$$B\cos 2\theta = (A - C)\sin 2\theta$$

Simplify.

$$\frac{B\cos 2\theta}{B\sin 2\theta} = \frac{(A - C)\sin 2\theta}{B\sin 2\theta}$$

Divide both sides by $B\sin 2\theta$.

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{A - C}{B}$$

Simplify.

$$\cot 2\theta = \frac{A - C}{B}$$

Apply a quotient identity: $\cot 2\theta = \dfrac{\cos 2\theta}{\sin 2\theta}$.

If $\cot 2\theta$ is positive, we will select θ so that $0° < \theta < 45°$. If $\cot 2\theta$ is negative, we will select θ so that $45° < \theta < 90°$. Thus θ, the angle of rotation, is always an acute angle.

Here is a step-by-step procedure for writing the equation of a rotated conic section in standard form:

Study Tip

What do you do after substituting the expressions for x and y from the rotation formulas into the given equation? You must simplify the resulting equation by expanding and collecting like terms. Work through this process slowly and carefully, allowing lots of room on your paper.

If your rotation equations are correct but you obtain an equation that has an $x'y'$-term, you have made an error in the algebraic simplification.

Writing the Equation of a Rotated Conic in Standard Form

1. Use the given equation
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0$$
to find $\cot 2\theta$.
$$\cot 2\theta = \frac{A - C}{B}$$
2. Use the expression for $\cot 2\theta$ to determine θ, the angle of rotation.
3. Substitute θ in the rotation formulas
$$x = x' \cos\theta - y' \sin\theta \quad \text{and} \quad y = x' \sin\theta + y' \cos\theta$$
and simplify.
4. Substitute the expressions for x and y from the rotation formulas in the given equation and simplify. The resulting equation should have no $x'y'$-term.
5. Write the equation involving x' and y' in standard form.

Using the equation in step 5, you can graph the conic section in the rotated $x'y'$-system.

EXAMPLE 3 Writing the Equation of a Rotated Conic Section in Standard Form

Rewrite the equation
$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$
in a rotated $x'y'$-system without an $x'y'$-term. Express the equation in the standard form of a conic section. Graph the conic section in the rotated system.

Solution

Step 1 Use the given equation to find $\cot 2\theta$. We need to identify the constants A, B, and C in the given equation.

$$7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$$

A is the coefficient of the x^2-term: $A = 7$.

B is the coefficient of the xy-term: $B = -6\sqrt{3}$.

C is the coefficient of the y^2-term: $C = 13$.

The appropriate angle θ through which to rotate the axes satisfies the equation

$$\cot 2\theta = \frac{A - C}{B} = \frac{7 - 13}{-6\sqrt{3}} = \frac{-6}{-6\sqrt{3}} = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}.$$

Step 2 Use the expression for $\cot 2\theta$ to determine the angle of rotation. We have $\cot 2\theta = \frac{\sqrt{3}}{3}$. Based on our knowledge of exact values for trigonometric functions, we conclude that $2\theta = 60°$. Thus, $\theta = 30°$.

Step 3 Substitute θ in the rotation formulas $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$ and simplify. Substituting 30° for θ,

$$x = x' \cos 30° - y' \sin 30° = x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) = \frac{\sqrt{3}x' - y'}{2}$$

$$y = x' \sin 30° + y' \cos 30° = x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right) = \frac{x' + \sqrt{3}y'}{2}.$$

Step 4 Substitute the expressions for x and y from the rotation formulas in the given equation and simplify.

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$ This is the given equation.

$$7\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 13\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0$$ Substitute the expressions for x and y from the rotation formulas.

$$7\left(\frac{3x'^2 - 2\sqrt{3}x'y' + y'^2}{4}\right) - 6\sqrt{3}\left(\frac{\sqrt{3}x'^2 + 3x'y' - x'y' - \sqrt{3}y'^2}{4}\right) + 13\left(\frac{x'^2 + 2\sqrt{3}x'y' + 3y'^2}{4}\right) - 16 = 0$$ Square and multiply.

$$7(3x'^2 - 2\sqrt{3}x'y' + y'^2) - 6\sqrt{3}(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2) + 13(x'^2 + 2\sqrt{3}x'y' + 3y'^2) - 64 = 0$$ Multiply both sides by 4.

$$21x'^2 - 14\sqrt{3}x'y' + 7y'^2 - 18x'^2 - 12\sqrt{3}x'y' + 18y'^2 + 13x'^2 + 26\sqrt{3}x'y' + 39y'^2 - 64 = 0$$ Distribute throughout parentheses.

$$21x'^2 - 18x'^2 + 13x'^2 - 14\sqrt{3}x'y' - 12\sqrt{3}x'y' + 26\sqrt{3}x'y' + 7y'^2 + 18y'^2 + 39y'^2 - 64 = 0$$ Rearrange terms.

$$16x'^2 + 64y'^2 - 64 = 0$$ Combine like terms.

Do you see how we "lost" the $x'y'$-term in the last equation?

$$-14\sqrt{3}x'y' - 12\sqrt{3}x'y' + 26\sqrt{3}x'y' = -26\sqrt{3}x'y' + 26\sqrt{3}x'y' = 0x'y' = 0$$

Step 5 Write the equation involving x' and y' in standard form. We can express $16x'^2 + 64y'^2 - 64 = 0$, an equation of an ellipse, in the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$16x'^2 + 64y'^2 - 64 = 0$$ This equation describes the ellipse relative to a system rotated through 30°.

$$16x'^2 + 64y'^2 = 64$$ Add 64 to both sides.

$$\frac{16x'^2}{64} + \frac{64y'^2}{64} = \frac{64}{64}$$ Divide both sides by 64.

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1$$ Simplify.

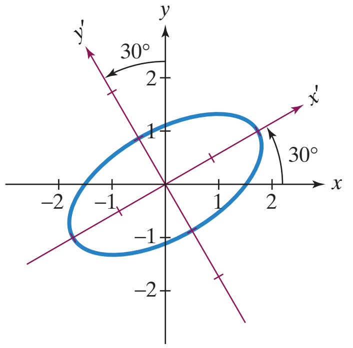


Figure 9.44 (repeated) The graph of $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$ or $\frac{x'^2}{4} - \frac{y'^2}{1} = 1$, a rotated ellipse

The equation $\frac{x'^2}{4} + \frac{y'^2}{1} = 1$ is the standard form of the equation of an ellipse. The major axis is on the x'-axis and the vertices are $(-2, 0)$ and $(2, 0)$. The minor axis is on the y'-axis with endpoints $(0, -1)$ and $(0, 1)$. The graph of the ellipse is shown in **Figure 9.44**. Does this graph look familiar? It should—you saw it earlier in this section on page 914.

Check Point 3 Rewrite the equation

$$2x^2 + \sqrt{3}xy + y^2 - 2 = 0$$

in a rotated $x'y'$-system without an $x'y'$-term. Express the equation in the standard form of a conic section. Graph the conic section in the rotated system.

Technology

In order to graph a general second-degree equation in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

using a graphing utility, it is necessary to solve for y. Rewrite the equation as a quadratic equation in y.

$$Cy^2 + (Bx + E)y + (Ax^2 + Dx + F) = 0$$

By applying the quadratic formula, the graph of this equation can be obtained by entering

$$y_1 = \frac{-(Bx + E) + \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C}$$

and

$$y_2 = \frac{-(Bx + E) - \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C}.$$

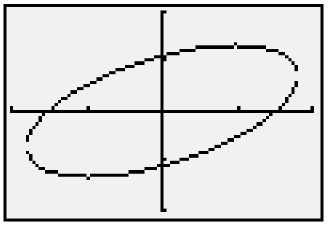
The graph of

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$

is shown on the right in a $[-2, 2, 1]$ by $[-2, 2, 1]$ viewing rectangle. The graph was obtained by entering the equations for y_1 and y_2 shown above with

$$A = 7, B = -6\sqrt{3}, C = 13, D = 0, E = 0,$$

$$\text{and } F = -16.$$



In Example 3 and Check Point 3, we found θ, the angle of rotation, directly because we recognized $\frac{\sqrt{3}}{3}$ as the value of $\cot 60°$. What do we do if $\cot 2\theta$ is not the cotangent of one of the familiar angles? We use $\cot 2\theta$ to find $\sin\theta$ and $\cos\theta$ as follows:

- Use a sketch of $\cot 2\theta$ to find $\cos 2\theta$.
- Find $\sin\theta$ and $\cos\theta$ using the identities

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} \qquad \text{and} \qquad \cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}}.$$

Because θ is an acute angle, the positive square roots are appropriate.

The resulting values for $\sin\theta$ and $\cos\theta$ are used to write the rotation formulas that give an equation with no $x'y'$-term.

EXAMPLE 4 Graphing the Equation of a Rotated Conic

Graph relative to a rotated $x'y'$-system in which the equation has no $x'y'$-term:

$$16x^2 - 24xy + 9y^2 + 110x - 20y + 100 = 0.$$

Solution

Step 1 Use the given equation to find cot 2*θ*. With $A = 16$, $B = -24$, and $C = 9$, we have

$$\cot 2\theta = \frac{A - C}{B} = \frac{16 - 9}{-24} = -\frac{7}{24}.$$

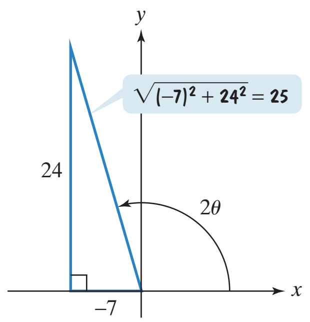


Figure 9.47 Using $\cot 2\theta$ to find $\cos 2\theta$

Step 2 Use the expression for cot 2*θ* to determine sin *θ* and cos *θ*. A rough sketch showing $\cot 2\theta$ is given in **Figure 9.47**. Because θ is always acute and $\cot 2\theta$ is negative, 2θ is in quadrant II. The third side of the triangle is found using $r = \sqrt{x^2 + y^2}$. Thus, $r = \sqrt{(-7)^2 + 24^2} = \sqrt{625} = 25$. By the definition of the cosine function,

$$\cos 2\theta = \frac{x}{r} = \frac{-7}{25} = -\frac{7}{25}.$$

Now we use identities to find values for $\sin\theta$ and $\cos\theta$.

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - \left(-\frac{7}{25}\right)}{2}}$$

$$= \sqrt{\frac{\frac{25}{25} + \frac{7}{25}}{2}} = \sqrt{\frac{\frac{32}{25}}{2}} = \sqrt{\frac{32}{50}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + \left(-\frac{7}{25}\right)}{2}}$$

$$= \sqrt{\frac{\frac{25}{25} - \frac{7}{25}}{2}} = \sqrt{\frac{\frac{18}{25}}{2}} = \sqrt{\frac{18}{50}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

Step 3 Substitute sin *θ* and cos *θ* in the rotation formulas

$$\boldsymbol{x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta}$$

and simplify. Substituting $\frac{4}{5}$ for $\sin\theta$ and $\frac{3}{5}$ for $\cos\theta$,

$$x = x'\left(\frac{3}{5}\right) - y'\left(\frac{4}{5}\right) = \frac{3x' - 4y'}{5}$$

$$y = x'\left(\frac{4}{5}\right) + y'\left(\frac{3}{5}\right) = \frac{4x' + 3y'}{5}.$$

Step 4 Substitute the expressions for *x* and *y* from the rotation formulas in the given equation and simplify.

$$16x^2 - 24xy + 9y^2 + 110x - 20y + 100 = 0$$

This is the given equation.

$$16\left(\frac{3x' - 4y'}{5}\right)^2 - 24\left(\frac{3x' - 4y'}{5}\right)\left(\frac{4x' + 3y'}{5}\right) + 9\left(\frac{4x' + 3y'}{5}\right)^2$$

$$+ 110\left(\frac{3x' - 4y'}{5}\right) - 20\left(\frac{4x' + 3y'}{5}\right) + 100 = 0$$

Substitute the expressions for x and y from the rotation formulas.

Work with the last equation at the bottom of the previous page. Take a few minutes to expand, multiply both sides of the equation by 25, and combine like terms. You should obtain

$$y'^2 + 2x' - 4y' + 4 = 0,$$

an equation that has no $x'y'$-term.

Step 5 Write the equation involving x' and y' in standard form. With only one variable that is squared, we have the equation of a parabola. We need to write the equation in the standard form $(y - k)^2 = 4p(x - h)$.

$y'^2 + 2x' - 4y' + 4 = 0$	This is the equation without an $x'y'$-term.
$y'^2 - 4y' = -2x' - 4$	Isolate the terms involving y'.
$y'^2 - 4y' + 4 = -2x' - 4 + 4$	Complete the square by adding the square of half the coefficient of y'.
$(y' - 2)^2 = -2x'$	Factor.

The standard form of the parabola's equation in the rotated $x'y'$-system is

$$(y' - 2)^2 = -2x'.$$

This is $(y' - k)^2$, with $k = 2$.
This is $4p$.
This is $x' - h$, with $h = 0$.

We see that $h = 0$ and $k = 2$. Thus, the vertex of the parabola in the $x'y'$-system is $(h, k) = (0, 2)$.

We can use the $x'y'$-system to graph the parabola. Using a calculator to solve $\sin\theta = \frac{4}{5}$, we find that $\theta = \sin^{-1}\frac{4}{5} \approx 53°$. Rotate the axes through approximately 53°. With $4p = -2$ and $p = -\frac{1}{2}$, the parabola's focus is $\frac{1}{2}$ unit to the left of the vertex, $(0, 2)$. Thus, the focus in the $x'y'$-system is $\left(-\frac{1}{2}, 2\right)$.

To graph the parabola, we use the vertex, $(0, 2)$, and the two endpoints of the latus rectum.

$$\text{length of latus rectum} = |4p| = |-2| = 2$$

The latus rectum extends 1 unit above and 1 unit below the focus, $\left(-\frac{1}{2}, 2\right)$. Thus, the endpoints of the latus rectum in the $x'y'$-system are $\left(-\frac{1}{2}, 3\right)$ and $\left(-\frac{1}{2}, 1\right)$. Using the rotated system, pass a smooth curve through the vertex and the two endpoints of the latus rectum. The graph of the parabola is shown in **Figure 9.48**.

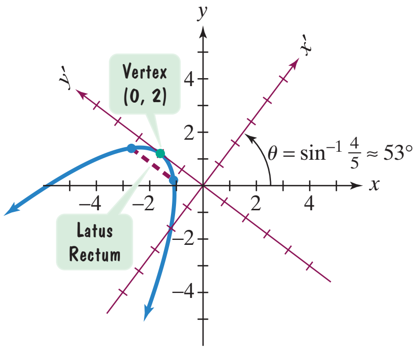


Figure 9.48 The graph of $(y' - 2)^2 = -2x'$ in a rotated $x'y'$-system

Check Point 4 Graph relative to a rotated $x'y'$-system in which the equation has no $x'y'$-term:

$$4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0.$$

4 Identify conics without rotating axes.

Identifying Conic Sections without Rotating Axes

We now know that the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0$$

can be rewritten as

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

in a rotated $x'y'$-system. A relationship between the coefficients of the two equations is given by

$$B^2 - 4AC = -4A'C'.$$

We also know that A' and C' can be used to identify the graph of the rotated equation. Thus, $B^2 - 4AC$ can also be used to identify the graph of the general second-degree equation.

Identifying a Conic Section without a Rotation of Axes

A nondegenerate conic section of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is

- a parabola if $B^2 - 4AC = 0$,
- an ellipse or a circle if $B^2 - 4AC < 0$, and
- a hyperbola if $B^2 - 4AC > 0$.

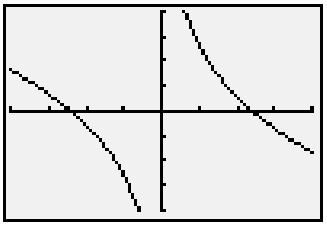
Technology

Graphic Connections

The graph of

$$11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$$

is shown in a $\left[-1, 1, \frac{1}{4}\right]$ by $\left[-1, 1, \frac{1}{4}\right]$ viewing rectangle. The graph verifies that the equation represents a rotated hyperbola.



EXAMPLE 5 Identifying a Conic Section without Rotating Axes

Identify the graph of

$$11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0.$$

Solution We use A, B, and C to identify the conic section.

$$11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$$

$A = 11$ $B = 10\sqrt{3}$ $C = 1$

$$B^2 - 4AC = (10\sqrt{3})^2 - 4(11)(1) = 100 \cdot 3 - 44 = 256 > 0$$

Because $B^2 - 4AC > 0$, the graph of the equation is a hyperbola.

Check Point 5 Identify the graph of $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$.

Exercise Set 9.4

Practice Exercises

In Exercises 1–8, identify each equation without completing the square.

1. $y^2 - 4x + 2y + 21 = 0$
2. $y^2 - 4x - 4y = 0$
3. $4x^2 - 9y^2 - 8x - 36y - 68 = 0$
4. $9x^2 + 25y^2 - 54x - 200y + 256 = 0$
5. $4x^2 + 4y^2 + 12x + 4y + 1 = 0$
6. $9x^2 + 4y^2 - 36x + 8y + 31 = 0$
7. $100x^2 - 7y^2 + 90y - 368 = 0$
8. $y^2 + 8x + 6y + 25 = 0$

In Exercises 9–14, write each equation in terms of a rotated $x'y'$-system using θ, the angle of rotation. Write the equation involving x' and y' in standard form.

9. $xy = -1$; $\theta = 45°$
10. $xy = -4$; $\theta = 45°$
11. $x^2 - 4xy + y^2 - 3 = 0$; $\theta = 45°$
12. $13x^2 - 10xy + 13y^2 - 72 = 0$; $\theta = 45°$
13. $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0$; $\theta = 30°$
14. $13x^2 - 6\sqrt{3}xy + 7y^2 - 16 = 0$; $\theta = 60°$

In Exercises 15–26, write the appropriate rotation formulas so that in a rotated system the equation has no $x'y'$-term.

15. $x^2 + xy + y^2 - 10 = 0$
16. $x^2 + 4xy + y^2 - 3 = 0$
17. $3x^2 - 10xy + 3y^2 - 32 = 0$
18. $5x^2 - 8xy + 5y^2 - 9 = 0$
19. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$
20. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$
21. $10x^2 + 24xy + 17y^2 - 9 = 0$
22. $32x^2 - 48xy + 18y^2 - 15x - 20y = 0$
23. $x^2 + 4xy - 2y^2 - 1 = 0$
24. $3xy - 4y^2 + 18 = 0$
25. $34x^2 - 24xy + 41y^2 - 25 = 0$
26. $6x^2 - 6xy + 14y^2 - 45 = 0$

In Exercises 27–38,

a. *Rewrite the equation in a rotated $x'y'$-system without an $x'y'$-term. Use the appropriate rotation formulas from Exercises 15–26.*

b. *Express the equation involving x' and y' in the standard form of a conic section.*

c. *Use the rotated system to graph the equation.*

27. $x^2 + xy + y^2 - 10 = 0$

28. $x^2 + 4xy + y^2 - 3 = 0$

29. $3x^2 - 10xy + 3y^2 - 32 = 0$

30. $5x^2 - 8xy + 5y^2 - 9 = 0$

31. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$

32. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$

33. $10x^2 + 24xy + 17y^2 - 9 = 0$

34. $32x^2 - 48xy + 18y^2 - 15x - 20y = 0$

35. $x^2 + 4xy - 2y^2 - 1 = 0$

36. $3xy - 4y^2 + 18 = 0$

37. $34x^2 - 24xy + 41y^2 - 25 = 0$

38. $6x^2 - 6xy + 14y^2 - 45 = 0$

In Exercises 39–44, identify each equation without applying a rotation of axes.

39. $5x^2 - 2xy + 5y^2 - 12 = 0$

40. $10x^2 + 24xy + 17y^2 - 9 = 0$

41. $24x^2 + 16\sqrt{3}xy + 8y^2 - x + \sqrt{3}y - 8 = 0$

42. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

43. $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0$

44. $4xy + 3y^2 + 4x + 6y - 1 = 0$

Practice Plus

In Exercises 45–48,

- *If the graph of the equation is an ellipse, find the coordinates of the vertices on the minor axis.*
- *If the graph of the equation is a hyperbola, find the equations of the asymptotes.*
- *If the graph of the equation is a parabola, find the coordinates of the vertex.*

Express answers relative to an $x'y'$-system in which the given equation has no $x'y'$-term. Assume that the $x'y'$-system has the same origin as the xy-system.

45. $5x^2 - 6xy + 5y^2 - 8 = 0$

46. $2x^2 - 4xy + 5y^2 - 36 = 0$

47. $x^2 - 4xy + 4y^2 + 5\sqrt{5}y - 10 = 0$

48. $x^2 + 4xy - 2y^2 - 6 = 0$

Writing in Mathematics

49. Explain how to identify the graph of

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

50. If there is a 60° angle from the positive x-axis to the positive x'-axis, explain how to obtain the rotation formulas for x and y.

51. How do you obtain the angle of rotation so that a general second-degree equation has no $x'y'$-term in a rotated $x'y'$-system?

52. What is the most time-consuming part in using a graphing utility to graph a general second-degree equation with an xy-term?

53. Explain how to identify the graph of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Technology Exercises

In Exercises 54–60, use a graphing utility to graph each equation.

54. $x^2 + 4xy + y^2 - 3 = 0$

55. $7x^2 + 8xy + y^2 - 1 = 0$

56. $3x^2 + 4xy + 6y^2 - 7 = 0$

57. $3x^2 - 6xy + 3y^2 + 10x - 8y - 2 = 0$

58. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$

59. $x^2 + 4xy + 4y^2 + 10\sqrt{5}x - 9 = 0$

60. $7x^2 + 6xy + 2.5y^2 - 14x + 4y + 9 = 0$

Critical Thinking Exercises

Make Sense? *In Exercises 61–64, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

61. I graphed $2x^2 - 3y^2 + 6y + 4 = 0$ by using the procedure for writing the equation of a rotated conic in standard form.

62. In order to graph an ellipse whose equation contained an xy-term, I used a rotated coordinate system that placed the ellipse's center at the origin.

63. Although the algebra of rotations can get ugly, the main idea is that rotation through an appropriate angle will transform a general second-degree equation into an equation in x' and y' without an $x'y'$-term.

64. I can verify that $2xy - 9 = 0$ is the equation of a hyperbola by rotating the axes through 45° or by showing that $B^2 - 4AC > 0$.

65. Explain the relationship between the graph of $3x^2 - 2xy + 3y^2 + 2 = 0$ and the sound made by one hand clapping. Begin by following the directions for Exercises 27–38. (You will first need to write rotation formulas that eliminate the $x'y'$-term.)

66. What happens to the equation $x^2 + y^2 = r^2$ in a rotated $x'y'$-system?

In Exercises 67–68, let

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

be an equation of a conic section in an xy-coordinate system. Let $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$ be the equation of the conic section in the rotated $x'y'$-coordinate system. Use the coefficients A', B', and C', shown in the equation with the voice balloon pointing to B' on page 917, to prove the following relationships.

67. $A' + C' = A + C$

68. $B'^2 - 4A'C' = B^2 - 4AC$

Group Exercise

69. Many public and private organizations and schools provide educational materials and information for the blind and visually impaired. Using your library, resources on the World Wide Web, or local organizations, investigate how your group or college could make a contribution to enhance the study of mathematics for the blind and visually impaired. In relation to conic sections, group members should discuss how to create graphs in tactile, or touchable, form that show blind students the visual structure of the conics, including asymptotes, intercepts, end behavior, and rotations.

Preview Exercises

Exercises 70–72 will help you prepare for the material covered in the next section. In each exercise, graph the equation in a rectangular coordinate system.

70. $y^2 = 4(x + 1)$

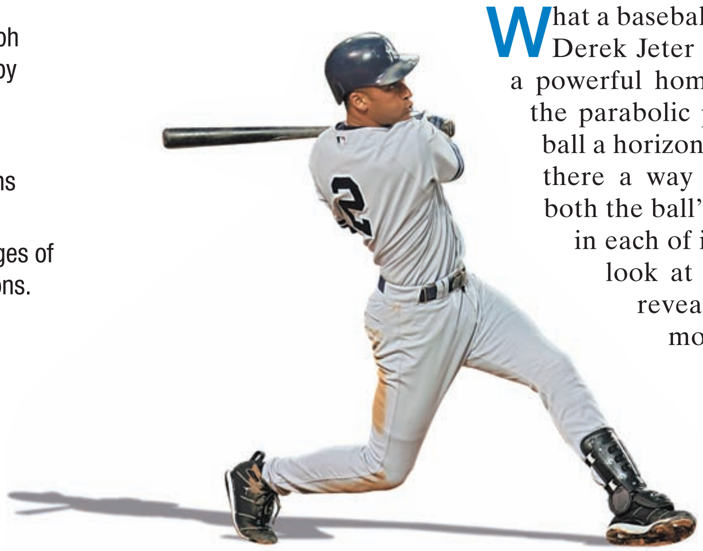
71. $y = \frac{1}{2}x^2 + 1, \quad x \geq 0$

72. $\frac{x^2}{25} + \frac{y^2}{4} = 1$

Section 9.5 Parametric Equations

Objectives

1. Use point plotting to graph plane curves described by parametric equations.
2. Eliminate the parameter.
3. Find parametric equations for functions.
4. Understand the advantages of parametric representations.



What a baseball game! You got to see the great Derek Jeter of the New York Yankees blast a powerful homer. In less than eight seconds, the parabolic path of his home run took the ball a horizontal distance of over 1000 feet. Is there a way to model this path that gives both the ball's location and the time that it is in each of its positions? In this section, we look at ways of describing curves that reveal the where and the when of motion.

Plane Curves and Parametric Equations

You throw a ball from a height of 6 feet, with an initial velocity of 90 feet per second and at an angle of 40° with the horizontal. After t seconds, the location of the ball can be described by

$$x = (90 \cos 40°)t \quad \text{and} \quad y = 6 + (90 \sin 40°)t - 16t^2.$$

This is the ball's horizontal distance, in feet.

This is the ball's vertical height, in feet.

Because we can use these equations to calculate the location of the ball at any time t, we can describe the path of the ball. For example, to determine the location when $t = 1$ second, substitute 1 for t in each equation:

$$x = (90 \cos 40°)t = (90 \cos 40°)(1) \approx 68.9 \text{ feet}$$

$$y = 6 + (90 \sin 40°)t - 16t^2 = 6 + (90 \sin 40°)(1) - 16(1)^2 \approx 47.9 \text{ feet.}$$

This tells us that after one second, the ball has traveled a horizontal distance of approximately 68.9 feet, and the height of the ball is approximately 47.9 feet. **Figure 9.49** on the next page displays this information and the results for calculations corresponding to $t = 2$ seconds and $t = 3$ seconds.