24. $(x^2 + 2)^2$

25. $\sqrt{50} \cdot \sqrt{6}$

26. $\dfrac{11}{7 - \sqrt{3}}$

27. $\dfrac{11}{\sqrt{3}}$

In Exercises 28–34, factor completely, or state that the polynomial is prime.

28. $7x^2 - 22x + 3$

29. $x^2 - 2x + 4$

30. $x^3 + 5x^2 + 3x + 15$

31. $3x^2 - 4xy - 7y^2$

32. $64y - y^4$

33. $50x^3 + 20x^2 + 2x$

34. $x^2 - 6x + 9 - 49y^2$

In Exercises 35–36, factor and simplify each algebraic expression.

35. $x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}} + x^{\frac{1}{2}}$

36. $(x^2 + 1)^{\frac{1}{2}} - 10(x^2 + 1)^{-\frac{1}{2}}$

37. List all the rational numbers in this set:

$$\left\{-11, -\frac{3}{7}, 0, 0.45, \sqrt{23}, \sqrt{25}\right\}.$$

In Exercises 38–39, rewrite each expression without absolute value bars.

38. $|2 - \sqrt{13}|$

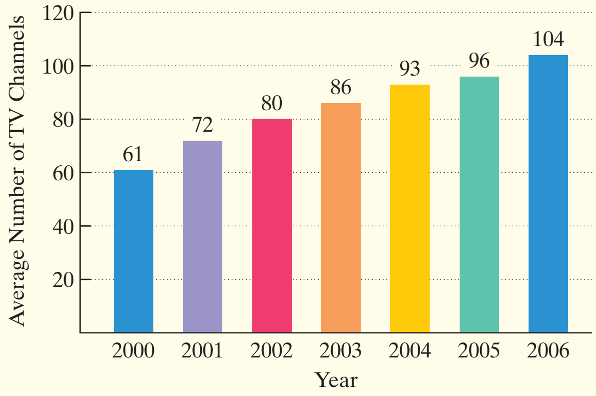
39. $x^2|x|$ if $x < 0$

40. If the population of the United States is approximately 3.0×10^8 and each person spends about \$140 per year on ice cream, express the total annual spending on ice cream in scientific notation.

41. A human brain contains 3×10^{10} neurons and a gorilla brain contains 7.5×10^9 neurons. How many times as many neurons are in the brain of a human as in the brain of a gorilla?

42. The number of TV channels is increasing. The bar graph shows the total channels available in the average U.S. household from 2000 through 2006.

Number of TV Channels in the Average United States Household



Source: Nielsen Media Research

Here are two mathematical models for the data shown by the graph. In each formula, N represents the number of TV channels in the average U.S. household x years after 2000.

Model 1 $N = 6.8x + 64$

Model 2 $N = -0.5x^2 + 9.5x + 62$

a. Which model best describes the data for 2000?

b. Does the polynomial model of degree 2 underestimate or overestimate the number of channels for 2006? By how many channels?

c. According to the polynomial model of degree 1, how many channels will the average household have in 2010?

Section P.6 Rational Expressions

Objectives

1. Specify numbers that must be excluded from the domain of a rational expression.
2. Simplify rational expressions.
3. Multiply rational expressions.
4. Divide rational expressions.
5. Add and subtract rational expressions.
6. Simplify complex rational expressions.
7. Simplify fractional expressions that occur in calculus.
8. Rationalize numerators.

How do we describe the costs of reducing environmental pollution? We often use algebraic expressions involving quotients of polynomials. For example, the algebraic expression

$$\frac{250x}{100 - x}$$

describes the cost, in millions of dollars, to remove x percent of the pollutants that are discharged into a river. Removing a modest percentage of pollutants, say 40%, is far less costly than removing a substantially greater percentage, such as 95%. We see this by evaluating the algebraic expression for $x = 40$ and $x = 95$.

$$\text{Evaluating } \frac{250x}{100-x} \text{ for}$$

$x = 40$: $\qquad\qquad x = 95$:

$$\text{Cost is } \frac{250(40)}{100-40} \approx 167. \qquad \text{Cost is } \frac{250(95)}{100-95} = 4750.$$

The cost increases from approximately \$167 million to a possibly prohibitive \$4750 million, or \$4.75 billion. Costs spiral upward as the percentage of removed pollutants increases.

Many algebraic expressions that describe costs of environmental projects are examples of *rational expressions*. First we will define rational expressions. Then we will review how to perform operations with such expressions.

Discovery

What happens if you try substituting 100 for x in

$$\frac{250x}{100-x}?$$

What does this tell you about the cost of cleaning up all of the river's pollutants?

1 Specify numbers that must be excluded from the domain of a rational expression.

Rational Expressions

A **rational expression** is the quotient of two polynomials. Some examples are

$$\frac{x-2}{4}, \quad \frac{4}{x-2}, \quad \frac{x}{x^2-1}, \quad \text{and} \quad \frac{x^2+1}{x^2+2x-3}.$$

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Because rational expressions indicate division and division by zero is undefined, **we must exclude numbers from a rational expression's domain that make the denominator zero**.

EXAMPLE 1 Excluding Numbers from the Domain

Find all the numbers that must be excluded from the domain of each rational expression:

a. $\dfrac{4}{x-2}$ **b.** $\dfrac{x}{x^2-1}$.

Solution To determine the numbers that must be excluded from each domain, examine the denominators.

a. $\dfrac{4}{x-2}$ — This denominator would equal zero if $x = 2$.

b. $\dfrac{x}{x^2-1} = \dfrac{x}{(x+1)(x-1)}$ — This factor ($x+1$) would equal zero if $x = -1$. This factor ($x-1$) would equal zero if $x = 1$.

For the rational expression in part (a), we must exclude 2 from the domain. For the rational expression in part (b), we must exclude both -1 and 1 from the domain. These excluded numbers are often written to the right of a rational expression:

$$\frac{4}{x-2},\ x \neq 2 \qquad \frac{x}{x^2-1},\ x \neq -1,\ x \neq 1$$

Check Point 1 Find all the numbers that must be excluded from the domain of each rational expression:

a. $\dfrac{7}{x+5}$ **b.** $\dfrac{x}{x^2-36}$.

2 Simplify rational expressions.

Simplifying Rational Expressions

A rational expression is **simplified** if its numerator and denominator have no common factors other than 1 or −1. The following procedure can be used to simplify rational expressions:

Simplifying Rational Expressions

1. Factor the numerator and the denominator completely.
2. Divide both the numerator and the denominator by any common factors.

EXAMPLE 2 Simplifying Rational Expressions

Simplify:

a. $\frac{x^3 + x^2}{x + 1}$ **b.** $\frac{x^2 + 6x + 5}{x^2 - 25}$.

Solution

a. $\frac{x^3 + x^2}{x + 1} = \frac{x^2(x + 1)}{x + 1}$ Factor the numerator. Because the denominator is $x + 1$, $x \neq -1$.

$= \frac{x^2\overset{1}{\cancel{(x + 1)}}}{\underset{1}{\cancel{x + 1}}}$ Divide out the common factor, $x + 1$.

$= x^2, x \neq -1$ Denominators of 1 need not be written because $\frac{a}{1} = a$.

b. $\frac{x^2 + 6x + 5}{x^2 - 25} = \frac{(x + 5)(x + 1)}{(x + 5)(x - 5)}$ Factor the numerator and denominator. Because the denominator is $(x + 5)(x - 5)$, $x \neq -5$ and $x \neq 5$.

$= \frac{\overset{1}{\cancel{(x + 5)}}(x + 1)}{\underset{1}{\cancel{(x + 5)}}(x - 5)}$ Divide out the common factor, $x + 5$.

$= \frac{x + 1}{x - 5}, \quad x \neq -5, \quad x \neq 5$

Check Point 2 Simplify:

a. $\frac{x^3 + 3x^2}{x + 3}$ **b.** $\frac{x^2 - 1}{x^2 + 2x + 1}$.

3 Multiply rational expressions.

Multiplying Rational Expressions

The product of two rational expressions is the product of their numerators divided by the product of their denominators. Here is a step-by-step procedure for multiplying rational expressions:

Multiplying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerators and multiply the remaining factors in the denominators.

EXAMPLE 3 Multiplying Rational Expressions

Multiply: $\frac{x-7}{x-1} \cdot \frac{x^2-1}{3x-21}$.

Solution

$$\frac{x-7}{x-1} \cdot \frac{x^2-1}{3x-21}$$

This is the given multiplication problem.

$$= \frac{x-7}{x-1} \cdot \frac{(x+1)(x-1)}{3(x-7)}$$

Factor as many numerators and denominators as possible. Because the denominators have factors of $x - 1$ and $x - 7$, $x \neq 1$ and $x \neq 7$.

$$= \frac{\overset{1}{\cancel{x-7}}}{\underset{1}{\cancel{x-1}}} \cdot \frac{(x+1)\overset{1}{\cancel{(x-1)}}}{3\underset{1}{\cancel{(x-7)}}}$$

Divide numerators and denominators by common factors.

$$= \frac{x+1}{3}, x \neq 1, x \neq 7$$

Multiply the remaining factors in the numerators and denominators.

These excluded numbers from the domain must also be excluded from the simplified expression's domain.

Check Point 3 Multiply:

$$\frac{x+3}{x^2-4} \cdot \frac{x^2-x-6}{x^2+6x+9}.$$

4 Divide rational expressions.

Dividing Rational Expressions

The quotient of two rational expressions is the product of the first expression and the multiplicative inverse, or reciprocal, of the second expression. The reciprocal is found by interchanging the numerator and the denominator. Thus, **we find the quotient of two rational expressions by inverting the divisor and multiplying**.

EXAMPLE 4 Dividing Rational Expressions

Divide: $\frac{x^2-2x-8}{x^2-9} \div \frac{x-4}{x+3}$.

Solution

$$\frac{x^2-2x-8}{x^2-9} \div \frac{x-4}{x+3}$$

This is the given division problem.

$$= \frac{x^2-2x-8}{x^2-9} \cdot \frac{x+3}{x-4}$$

Invert the divisor and multiply.

$$= \frac{(x-4)(x+2)}{(x+3)(x-3)} \cdot \frac{x+3}{x-4}$$

Factor as many numerators and denominators as possible. For nonzero denominators, $x \neq -3$, $x \neq 3$, and $x \neq 4$.

$$= \frac{\overset{1}{\cancel{(x-4)}}(x+2)}{\underset{1}{\cancel{(x+3)}}(x-3)} \cdot \frac{\overset{1}{\cancel{(x+3)}}}{\underset{1}{\cancel{(x-4)}}}$$

Divide numerators and denominators by common factors.

$$= \frac{x+2}{x-3}, x \neq -3, x \neq 3, x \neq 4$$

Multiply the remaining factors in the numerators and in the denominators.

Check Point 4 Divide:

$$\frac{x^2 - 2x + 1}{x^3 + x} \div \frac{x^2 + x - 2}{3x^2 + 3}.$$

5 Add and subtract rational expressions.

Adding and Subtracting Rational Expressions with the Same Denominator

We add or subtract rational expressions with the same denominator by (1) adding or subtracting the numerators, (2) placing this result over the common denominator, and (3) simplifying, if possible.

EXAMPLE 5 Subtracting Rational Expressions with the Same Denominator

Subtract: $\frac{5x + 1}{x^2 - 9} - \frac{4x - 2}{x^2 - 9}$.

Solution

Don't forget the parentheses.

$$\frac{5x + 1}{x^2 - 9} - \frac{4x - 2}{x^2 - 9} = \frac{5x + 1 - (4x - 2)}{x^2 - 9}$$

Subtract numerators and include parentheses to indicate that both terms are subtracted. Place this difference over the common denominator.

$$= \frac{5x + 1 - 4x + 2}{x^2 - 9}$$

Remove parentheses and then change the sign of each term in parentheses.

$$= \frac{x + 3}{x^2 - 9}$$

Combine like terms.

$$= \frac{\overset{1}{\cancel{x + 3}}}{\underset{1}{\cancel{(x + 3)}}(x - 3)}$$

Factor and simplify ($x \neq -3$ and $x \neq 3$).

$$= \frac{1}{x - 3}, x \neq -3, x \neq 3$$

Study Tip

Example 5 shows that when a numerator is being subtracted, we must subtract every term in that expression.

Check Point 5 Subtract: $\frac{x}{x + 1} - \frac{3x + 2}{x + 1}$.

Adding and Subtracting Rational Expressions with Different Denominators

Rational expressions that have no common factors in their denominators can be added or subtracted using one of the following properties:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}, b \neq 0, d \neq 0.$$

The denominator, bd, is the product of the factors in the two denominators. Because we are considering rational expressions that have no common factors in their denominators, the product bd gives the least common denominator.

EXAMPLE 6 Subtracting Rational Expressions Having No Common Factors in Their Denominators

Subtract: $\dfrac{x+2}{2x-3} - \dfrac{4}{x+3}$.

Solution We need to find the least common denominator. This is the product of the distinct factors in each denominator, namely $(2x-3)(x+3)$. We can therefore use the subtraction property given previously as follows:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

$$\frac{x+2}{2x-3} - \frac{4}{x+3} = \frac{(x+2)(x+3)-(2x-3)4}{(2x-3)(x+3)}$$

Observe that $a = x+2$, $b = 2x-3$, $c = 4$, and $d = x+3$.

$$= \frac{x^2+5x+6-(8x-12)}{(2x-3)(x+3)}$$

Multiply.

$$= \frac{x^2+5x+6-8x+12}{(2x-3)(x+3)}$$

Remove parentheses and then change the sign of each term in parentheses.

$$= \frac{x^2-3x+18}{(2x-3)(x+3)}, \; x \neq \frac{3}{2}, \; x \neq -3$$

Combine like terms in the numerator.

Check Point 6 Add: $\dfrac{3}{x+1} + \dfrac{5}{x-1}$.

The **least common denominator**, or LCD, of several rational expressions is a polynomial consisting of the product of all prime factors in the denominators, with each factor raised to the greatest power of its occurrence in any denominator. When adding and subtracting rational expressions that have different denominators with one or more common factors in the denominators, it is efficient to find the least common denominator first.

Finding the Least Common Denominator

1. Factor each denominator completely.
2. List the factors of the first denominator.
3. Add to the list in step 2 any factors of the second denominator that do not appear in the list.
4. Form the product of the factors from the list in step 3. This product is the least common denominator.

EXAMPLE 7 Finding the Least Common Denominator

Find the least common denominator of

$$\frac{7}{5x^2+15x} \quad \text{and} \quad \frac{9}{x^2+6x+9}.$$

$$\frac{7}{5x^2 + 15x} \quad \text{and} \quad \frac{9}{x^2 + 6x + 9}$$

The given rational expressions (repeated)

Solution

Step 1 Factor each denominator completely.

$$5x^2 + 15x = 5x(x + 3)$$
$$x^2 + 6x + 9 = (x + 3)^2 \quad \text{or} \quad (x + 3)(x + 3)$$

Factors are 5, x, and $x + 3$. $\quad \frac{7}{5x^2 + 15x} \qquad \frac{9}{x^2 + 6x + 9}$ Factors are $x + 3$ and $x + 3$.

Step 2 List the factors of the first denominator.

$$5, x, x + 3$$

Step 3 Add any unlisted factors from the second denominator. One factor of $x^2 + 6x + 9$ is already in our list. That factor is $x + 3$. However, the other factor of $x + 3$ is not listed in step 2. We add a second factor of $x + 3$ to the list. We have

$$5, x, x + 3, x + 3.$$

Step 4 The least common denominator is the product of all factors in the final list. Thus,

$$5x(x + 3)(x + 3) \quad \text{or} \quad 5x(x + 3)^2$$

is the least common denominator.

Check Point 7 Find the least common denominator of

$$\frac{3}{x^2 - 6x + 9} \quad \text{and} \quad \frac{7}{x^2 - 9}.$$

Finding the least common denominator for two (or more) rational expressions is the first step needed to add or subtract the expressions.

Adding and Subtracting Rational Expressions That Have Different Denominators

1. Find the LCD of the rational expressions.
2. Rewrite each rational expression as an equivalent expression whose denominator is the LCD. To do so, multiply the numerator and the denominator of each rational expression by any factor(s) needed to convert the denominator into the LCD.
3. Add or subtract numerators, placing the resulting expression over the LCD.
4. If possible, simplify the resulting rational expression.

EXAMPLE 8 Adding Rational Expressions with Different Denominators

Add: $\dfrac{x + 3}{x^2 + x - 2} + \dfrac{2}{x^2 - 1}$.

Solution

Step 1 Find the least common denominator. Start by factoring the denominators.

$$x^2 + x - 2 = (x + 2)(x - 1)$$
$$x^2 - 1 = (x + 1)(x - 1)$$

The factors of the first denominator are $x + 2$ and $x - 1$. The only factor from the second denominator that is not listed is $x + 1$. Thus, the least common denominator is

$$(x + 2)(x - 1)(x + 1).$$

Step 2 Write equivalent expressions with the LCD as denominators. We must rewrite each rational expression with a denominator of $(x + 2)(x - 1)(x + 1)$. We do so by multiplying both the numerator and the denominator of each rational expression by any factor(s) needed to convert the expression's denominator into the LCD.

$$\frac{x+3}{(x+2)(x-1)} \cdot \frac{x+1}{x+1} = \frac{(x+3)(x+1)}{(x+2)(x-1)(x+1)}$$

Multiply the numerator and denominator by $x + 1$ to get $(x + 2)(x - 1)(x + 1)$, the LCD.

$$\frac{2}{(x+1)(x-1)} \cdot \frac{x+2}{x+2} = \frac{2(x+2)}{(x+2)(x-1)(x+1)}$$

Multiply the numerator and denominator by $x + 2$ to get $(x + 2)(x - 1)(x + 1)$, the LCD.

Because $\frac{x+1}{x+1} = 1$ and $\frac{x+2}{x+2} = 1$, we are not changing the value of either rational expression, only its appearance.

Now we are ready to perform the indicated addition.

$$\frac{x+3}{x^2+x-2} + \frac{2}{x^2-1}$$ This is the given problem.

$$= \frac{x+3}{(x+2)(x-1)} + \frac{2}{(x+1)(x-1)}$$ Factor the denominators. The LCD is $(x + 2)(x - 1)(x + 1)$.

$$= \frac{(x+3)(x+1)}{(x+2)(x-1)(x+1)} + \frac{2(x+2)}{(x+2)(x-1)(x+1)}$$ Rewrite equivalent expressions with the LCD.

Step 3 Add numerators, putting this sum over the LCD.

$$= \frac{(x+3)(x+1) + 2(x+2)}{(x+2)(x-1)(x+1)}$$

$$= \frac{x^2 + 4x + 3 + 2x + 4}{(x+2)(x-1)(x+1)}$$ Perform the multiplications in the numerator.

$$= \frac{x^2 + 6x + 7}{(x+2)(x-1)(x+1)}, \; x \neq -2, x \neq 1, x \neq -1$$ Combine like terms in the numerator: $4x + 2x = 6x$ and $3 + 4 = 7$.

Step 4 If necessary, simplify. Because the numerator is prime, no further simplification is possible.

Check Point 8 Subtract: $\frac{x}{x^2 - 10x + 25} - \frac{x-4}{2x-10}$.

6 Simplify complex rational expressions.

Complex Rational Expressions

Complex rational expressions, also called **complex fractions**, have numerators or denominators containing one or more rational expressions. Here are two examples of such expressions:

$$\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$$ Separate rational expressions occur in the numerator and the denominator.

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}.$$ Separate rational expressions occur in the numerator.

One method for simplifying a complex rational expression is to combine its numerator into a single expression and combine its denominator into a single expression. Then perform the division by inverting the denominator and multiplying.

EXAMPLE 9 Simplifying a Complex Rational Expression

Simplify: $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$.

Solution

Step 1 Add to get a single rational expression in the numerator.

$$1+\frac{1}{x}=\frac{1}{1}+\frac{1}{x}=\frac{1\cdot x}{1\cdot x}+\frac{1}{x}=\frac{x}{x}+\frac{1}{x}=\frac{x+1}{x}$$

The LCD is $1 \cdot x$, or x.

Step 2 Subtract to get a single rational expression in the denominator.

$$1-\frac{1}{x}=\frac{1}{1}-\frac{1}{x}=\frac{1\cdot x}{1\cdot x}-\frac{1}{x}=\frac{x}{x}-\frac{1}{x}=\frac{x-1}{x}$$

The LCD is $1 \cdot x$, or x.

Step 3 Perform the division indicated by the main fraction bar: Invert and multiply. If possible, simplify.

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x}\cdot\frac{x}{x-1}=\frac{x+1}{\overset{}{\cancel{x}}_{1}}\cdot\frac{\overset{1}{\cancel{x}}}{x-1}=\frac{x+1}{x-1}$$

Invert and multiply.

Check Point 9 Simplify: $\dfrac{\frac{1}{x}-\frac{3}{2}}{\frac{1}{x}+\frac{3}{4}}$.

A second method for simplifying a complex rational expression is to find the least common denominator of all the rational expressions in its numerator and denominator. Then multiply each term in its numerator and denominator by this least common denominator. Because we are multiplying by a form of 1, we will obtain an equivalent expression that does not contain fractions in its numerator or denominator. Here we use this method to simplify the complex rational expression in Example 9.

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\left(1+\frac{1}{x}\right)}{\left(1-\frac{1}{x}\right)}\cdot\frac{x}{x}$$

The least common denominator of all the rational expressions is x. Multiply the numerator and denominator by x. Because $\frac{x}{x}=1$, we are not changing the complex fraction $(x \neq 0)$.

$$=\frac{1\cdot x+\frac{1}{x}\cdot x}{1\cdot x-\frac{1}{x}\cdot x}$$

Use the distributive property. Be sure to distribute x to every term.

$$=\frac{x+1}{x-1},\ x\neq 0,\ x\neq 1$$

Multiply. The complex rational expression is now simplified.

EXAMPLE 10 Simplifying a Complex Rational Expression

Simplify: $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$.

Solution We will use the method of multiplying each of the three terms, $\frac{1}{x+h}$, $\frac{1}{x}$, and h, by the least common denominator. The least common denominator is $x(x+h)$.

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\frac{\left(\frac{1}{x+h}-\frac{1}{x}\right)x(x+h)}{hx(x+h)}$$ Multiply the numerator and denominator by $x(x+h)$, $h \neq 0$, $x \neq 0$, $x \neq -h$.

$$=\frac{\frac{1}{x+h}\cdot x(x+h)-\frac{1}{x}\cdot x(x+h)}{hx(x+h)}$$ Use the distributive property in the numerator.

$$=\frac{x-(x+h)}{hx(x+h)}$$ Simplify: $\frac{1}{x+h}x(x+h)=x$ and $\frac{1}{x}\cdot x(x+h)=x+h$.

$$=\frac{x-x-h}{hx(x+h)}$$ Subtract in the numerator. Remove parentheses and change the sign of each term in parentheses.

$$=\frac{-h}{hx(x+h)}$$ Simplify: $x-x-h=-h$.

$$=-\frac{1}{x(x+h)},\ h \neq 0, x \neq 0, x \neq -h$$ Divide the numerator and denominator by h.

Check Point 10 Simplify: $\dfrac{\dfrac{1}{x+7}-\dfrac{1}{x}}{7}$.

7 Simplify fractional expressions that occur in calculus.

Fractional Expressions in Calculus

Fractional expressions containing radicals occur frequently in calculus. Because of the radicals, these expressions are not rational expressions. However, they can often be simplified using the procedure for simplifying complex rational expressions.

EXAMPLE 11 Simplifying a Fractional Expression Containing Radicals

Simplify: $\dfrac{\sqrt{9-x^2}+\dfrac{x^2}{\sqrt{9-x^2}}}{9-x^2}$.

Solution

$$\frac{\sqrt{9-x^2}+\dfrac{x^2}{\sqrt{9-x^2}}}{9-x^2}$$

The least common denominator is $\sqrt{9-x^2}$.

$$=\frac{\sqrt{9-x^2}+\dfrac{x^2}{\sqrt{9-x^2}}}{9-x^2}\cdot\frac{\sqrt{9-x^2}}{\sqrt{9-x^2}}$$

Multiply the numerator and the denominator by $\sqrt{9-x^2}$.

$$=\frac{\sqrt{9-x^2}\sqrt{9-x^2}+\dfrac{x^2}{\sqrt{9-x^2}}\sqrt{9-x^2}}{(9-x^2)\sqrt{9-x^2}}$$

Use the distributive property in the numerator.

$$=\frac{(9-x^2)+x^2}{(9-x^2)^{\frac{3}{2}}}$$

In the denominator:
$$(9-x^2)^1(9-x^2)^{\frac{1}{2}}=(9-x^2)^{1+\frac{1}{2}}=(9-x^2)^{\frac{3}{2}}.$$

$$=\frac{9}{\sqrt{(9-x^2)^3}}$$

Because the original expression was in radical form, write the denominator in radical form.

Check Point 11 Simplify: $\dfrac{\sqrt{x}+\dfrac{1}{\sqrt{x}}}{x}$.

---

8 Rationalize numerators.

Another fractional expression that you will encounter in calculus is

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

Can you see that this expression is not defined if $h = 0$? However, in calculus, you will ask the following question:

> What happens to the expression as h takes on values that get closer and closer to 0, such as $h = 0.1$, $h = 0.01$, $h = 0.001$, $h = 0.0001$, and so on?

The question is answered by first **rationalizing the numerator**. This process involves rewriting the fractional expression as an equivalent expression in which the numerator no longer contains any radicals. **To rationalize a numerator, multiply by 1 to eliminate the radicals in the *numerator*. Multiply the numerator and the denominator by the conjugate of the numerator.**

EXAMPLE 12 Rationalizing a Numerator

Rationalize the numerator: $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

Solution The conjugate of the numerator is $\sqrt{x+h}+\sqrt{x}$. If we multiply the numerator and denominator by $\sqrt{x+h}+\sqrt{x}$, the simplified numerator will not contain a radical. Therefore, we multiply by 1, choosing $\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$ for 1.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$ Multiply by 1.

$$= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$ $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right) = \left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2$

$$= \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$ $\left(\sqrt{x+h}\right)^2 = x+h$ and $\left(\sqrt{x}\right)^2 = x$.

$$= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$ Simplify: $x+h-x=h$.

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}},\quad h\neq 0$$ Divide both the numerator and denominator by h.

Calculus Preview

In calculus, you will summarize the discussion on the right using the special notation

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{2\sqrt{x}}.$$

This is read "the limit of $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ as h approaches 0 equals $\frac{1}{2\sqrt{x}}$." Limits are discussed in Chapter 11, where we present an introduction to calculus.

What happens to $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ as h gets closer and closer to 0? In Example 12, we showed that

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

As h gets closer to 0, the expression on the right gets closer to $\frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}}$, or $\frac{1}{2\sqrt{x}}$. Thus, the fractional expression $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ approaches $\frac{1}{2\sqrt{x}}$ as h gets closer to 0.

Check Point 12 Rationalize the numerator: $\frac{\sqrt{x+3}-\sqrt{x}}{3}$.

Exercise Set P.6

Practice Exercises

In Exercises 1–6, find all numbers that must be excluded from the domain of each rational expression.

1. $\frac{7}{x-3}$ **2.** $\frac{13}{x+9}$ **3.** $\frac{x+5}{x^2-25}$

4. $\frac{x+7}{x^2-49}$ **5.** $\frac{x-1}{x^2+11x+10}$ **6.** $\frac{x-3}{x^2+4x-45}$

In Exercises 7–14, simplify each rational expression. Find all numbers that must be excluded from the domain of the simplified rational expression.

7. $\frac{3x-9}{x^2-6x+9}$ **8.** $\frac{4x-8}{x^2-4x+4}$

9. $\frac{x^2-12x+36}{4x-24}$ **10.** $\frac{x^2-8x+16}{3x-12}$

11. $\frac{y^2+7y-18}{y^2-3y+2}$ **12.** $\frac{y^2-4y-5}{y^2+5y+4}$

13. $\frac{x^2+12x+36}{x^2-36}$ **14.** $\frac{x^2-14x+49}{x^2-49}$

In Exercises 15–32, multiply or divide as indicated.

15. $\frac{x-2}{3x+9}\cdot\frac{2x+6}{2x-4}$ **16.** $\frac{6x+9}{3x-15}\cdot\frac{x-5}{4x+6}$

17. $\frac{x^2-9}{x^2}\cdot\frac{x^2-3x}{x^2+x-12}$ **18.** $\frac{x^2-4}{x^2-4x+4}\cdot\frac{2x-4}{x+2}$

19. $\frac{x^2-5x+6}{x^2-2x-3}\cdot\frac{x^2-1}{x^2-4}$ **20.** $\frac{x^2+5x+6}{x^2+x-6}\cdot\frac{x^2-9}{x^2-x-6}$

21. $\frac{x^3-8}{x^2-4}\cdot\frac{x+2}{3x}$ **22.** $\frac{x^2+6x+9}{x^3+27}\cdot\frac{1}{x+3}$

23. $\frac{x+1}{3}\div\frac{3x+3}{7}$ **24.** $\frac{x+5}{7}\div\frac{4x+20}{9}$

25. $\frac{x^2-4}{x}\div\frac{x+2}{x-2}$ **26.** $\frac{x^2-4}{x-2}\div\frac{x+2}{4x-8}$

27. $\frac{4x^2+10}{x-3}\div\frac{6x^2+15}{x^2-9}$ **28.** $\frac{x^2+x}{x^2-4}\div\frac{x^2-1}{x^2+5x+6}$

29. $\frac{x^2-25}{2x-2}\div\frac{x^2+10x+25}{x^2+4x-5}$

30. $\frac{x^2-4}{x^2+3x-10}\div\frac{x^2+5x+6}{x^2+8x+15}$

31. $\frac{x^2+x-12}{x^2+x-30}\cdot\frac{x^2+5x+6}{x^2-2x-3}\div\frac{x+3}{x^2+7x+6}$

32. $\frac{x^3-25x}{4x^2}\cdot\frac{2x^2-2}{x^2-6x+5}\div\frac{x^2+5x}{7x+7}$

In Exercises 33–58, add or subtract as indicated.

33. $\frac{4x+1}{6x+5}+\frac{8x+9}{6x+5}$

34. $\frac{3x+2}{3x+4}+\frac{3x+6}{3x+4}$

35. $\frac{x^2-2x}{x^2+3x}+\frac{x^2+x}{x^2+3x}$

36. $\frac{x^2-4x}{x^2-x-6}+\frac{4x-4}{x^2-x-6}$

37. $\frac{4x-10}{x-2}-\frac{x-4}{x-2}$

38. $\frac{2x+3}{3x-6}-\frac{3-x}{3x-6}$

39. $\frac{x^2+3x}{x^2+x-12}-\frac{x^2-12}{x^2+x-12}$

40. $\frac{x^2-4x}{x^2-x-6}-\frac{x-6}{x^2-x-6}$

41. $\frac{3}{x+4}+\frac{6}{x+5}$

42. $\frac{8}{x-2}+\frac{2}{x-3}$

43. $\frac{3}{x+1}-\frac{3}{x}$

44. $\frac{4}{x}-\frac{3}{x+3}$

45. $\frac{2x}{x+2}+\frac{x+2}{x-2}$

46. $\frac{3x}{x-3}-\frac{x+4}{x+2}$

47. $\frac{x+5}{x-5}+\frac{x-5}{x+5}$

48. $\frac{x+3}{x-3}+\frac{x-3}{x+3}$

49. $\frac{3}{2x+4}+\frac{2}{3x+6}$

50. $\frac{5}{2x+8}+\frac{7}{3x+12}$

51. $\frac{4}{x^2+6x+9}+\frac{4}{x+3}$

52. $\frac{3}{5x+2}+\frac{5x}{25x^2-4}$

53. $\frac{3x}{x^2+3x-10}-\frac{2x}{x^2+x-6}$

54. $\frac{x}{x^2-2x-24}-\frac{x}{x^2-7x+6}$

55. $\frac{x+3}{x^2-1}-\frac{x+2}{x-1}$

56. $\frac{x+5}{x^2-4}-\frac{x+1}{x-2}$

57. $\frac{4x^2+x-6}{x^2+3x+2}-\frac{3x}{x+1}+\frac{5}{x+2}$

58. $\frac{6x^2+17x-40}{x^2+x-20}+\frac{3}{x-4}-\frac{5x}{x+5}$

In Exercises 59–72, simplify each complex rational expression.

59. $\dfrac{\frac{x}{3}-1}{x-3}$

60. $\dfrac{\frac{x}{4}-1}{x-4}$

61. $\dfrac{1+\frac{1}{x}}{3-\frac{1}{x}}$

62. $\dfrac{8+\frac{1}{x}}{4-\frac{1}{x}}$

63. $\dfrac{\frac{1}{x}+\frac{1}{y}}{x+y}$

64. $\dfrac{1-\frac{1}{x}}{xy}$

65. $\dfrac{x-\frac{x}{x+3}}{x+2}$

66. $\dfrac{x-3}{x-\frac{3}{x-2}}$

67. $\dfrac{\frac{3}{x-2}-\frac{4}{x+2}}{\frac{7}{x^2-4}}$

68. $\dfrac{\frac{x}{x-2}+1}{\frac{3}{x^2-4}+1}$

69. $\dfrac{\frac{1}{x+1}}{\frac{1}{x^2-2x-3}+\frac{1}{x-3}}$

70. $\dfrac{\frac{6}{x^2+2x-15}-\frac{1}{x-3}}{\frac{1}{x+5}+1}$

71. $\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

72. $\dfrac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h}$

Exercises 73–78 contain fractional expressions that occur frequently in calculus. Simplify each expression.

73. $\dfrac{\sqrt{x}-\frac{1}{3\sqrt{x}}}{\sqrt{x}}$

74. $\dfrac{\sqrt{x}-\frac{1}{4\sqrt{x}}}{\sqrt{x}}$

75. $\dfrac{\frac{x^2}{\sqrt{x^2+2}}-\sqrt{x^2+2}}{x^2}$

76. $\dfrac{\sqrt{5-x^2}+\frac{x^2}{\sqrt{5-x^2}}}{5-x^2}$

77. $\dfrac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$

78. $\dfrac{\frac{1}{\sqrt{x+3}}-\frac{1}{\sqrt{x}}}{3}$

In Exercises 79–82, rationalize the numerator.

79. $\frac{\sqrt{x+5}-\sqrt{x}}{5}$

80. $\frac{\sqrt{x+7}-\sqrt{x}}{7}$

81. $\frac{\sqrt{x}+\sqrt{y}}{x^2-y^2}$

82. $\frac{\sqrt{x}-\sqrt{y}}{x^2-y^2}$

Practice Plus

In Exercises 83–90, perform the indicated operations. Simplify the result, if possible.

83. $\left(\frac{2x+3}{x+1}\cdot\frac{x^2+4x-5}{2x^2+x-3}\right)-\frac{2}{x+2}$

84. $\frac{1}{x^2-2x-8}\div\left(\frac{1}{x-4}-\frac{1}{x+2}\right)$

85. $\left(2-\frac{6}{x+1}\right)\left(1+\frac{3}{x-2}\right)$

86. $\left(4-\frac{3}{x+2}\right)\left(1+\frac{5}{x-1}\right)$

87. $\frac{y^{-1}-(y+5)^{-1}}{5}$

88. $\frac{y^{-1}-(y+2)^{-1}}{2}$

89. $\left(\frac{1}{a^3-b^3}\cdot\frac{ac+ad-bc-bd}{1}\right)-\frac{c-d}{a^2+ab+b^2}$

90. $\frac{ab}{a^2+ab+b^2}+\left(\frac{ac-ad-bc+bd}{ac-ad+bc-bd}\div\frac{a^3-b^3}{a^3+b^3}\right)$

Application Exercises

91. The rational expression

$$\frac{130x}{100-x}$$

describes the cost, in millions of dollars, to inoculate x percent of the population against a particular strain of flu.

a. Evaluate the expression for $x = 40$, $x = 80$, and $x = 90$. Describe the meaning of each evaluation in terms of percentage inoculated and cost.

b. For what value of x is the expression undefined?

c. What happens to the cost as x approaches 100%? How can you interpret this observation?

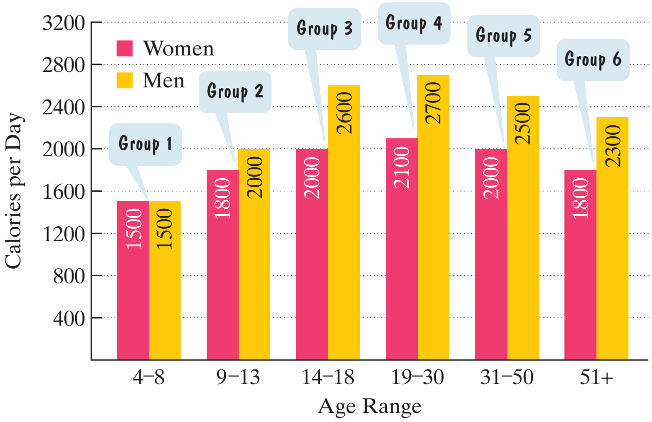
92. The average rate on a round-trip commute having a one-way distance d is given by the complex rational expression

$$\frac{2d}{\frac{d}{r_1}+\frac{d}{r_2}},$$

in which r_1 and r_2 are the average rates on the outgoing and return trips, respectively. Simplify the expression. Then find your average rate if you drive to campus averaging 40 miles per hour and return home on the same route averaging 30 miles per hour. Explain why the answer is not 35 miles per hour.

93. The bar graph shows the estimated number of calories per day needed to maintain energy balance for various gender and age groups for moderately active lifestyles. (Moderately active means a lifestyle that includes physical activity equivalent to walking 1.5 to 3 miles per day at 3 to 4 miles per hour, in addition to the light physical activity associated with typical day-to-day life.)

Calories Needed to Maintain Energy Balance for Moderately Active Lifestyles



Source: U.S.D.A.

a. The mathematical model

$$W = -66x^2 + 526x + 1030$$

describes the number of calories needed per day, W, by women in age group x with moderately active lifestyles. According to the model, how many calories per day are needed by women between the ages of 19 and 30, inclusive, with this lifestyle? Does this underestimate or overestimate the number shown by the graph? By how much?

b. The mathematical model

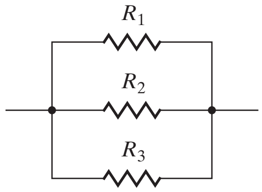
$$M = -120x^2 + 998x + 590$$

describes the number of calories needed per day, M, by men in age group x with moderately active lifestyles. According to the model, how many calories per day are needed by men between the ages of 19 and 30, inclusive, with this lifestyle? Does this underestimate or overestimate the number shown by the graph? By how much?

c. Write a simplified rational expression that describes the ratio of the number of calories needed per day by women in age group x to the number of calories needed per day by men in age group x for people with moderately active lifestyles.

94. If three resistors with resistances R_1, R_2, and R_3 are connected in parallel, their combined resistance is given by the expression

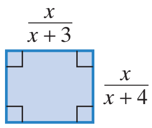
$$\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}.$$



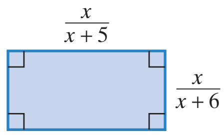
Simplify the complex rational expression. Then find the combined resistance when R_1 is 4 ohms, R_2 is 8 ohms, and R_3 is 12 ohms.

In Exercises 95–96, express the perimeter of each rectangle as a single rational expression.

95.



96.



Writing in Mathematics

97. What is a rational expression?

98. Explain how to determine which numbers must be excluded from the domain of a rational expression.

99. Explain how to simplify a rational expression.

100. Explain how to multiply rational expressions.

101. Explain how to divide rational expressions.

102. Explain how to add or subtract rational expressions with the same denominators.

103. Explain how to add rational expressions having no common factors in their denominators. Use $\frac{3}{x+5}+\frac{7}{x+2}$ in your explanation.

104. Explain how to find the least common denominator for denominators of $x^2 - 100$ and $x^2 - 20x + 100$.

105. Describe two ways to simplify $\dfrac{\frac{3}{x}+\frac{2}{x^2}}{\frac{1}{x^2}+\frac{2}{x}}$.

Explain the error in Exercises 106–108. Then rewrite the right side of the equation to correct the error that now exists.

106. $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$

107. $\frac{1}{x}+7=\frac{1}{x+7}$

108. $\frac{a}{x}+\frac{a}{b}=\frac{a}{x+b}$

Critical Thinking Exercises

Make Sense? *In Exercises 109–112, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

109. I evaluated $\frac{3x-3}{4x(x-1)}$ for $x = 1$ and obtained 0.

110. The rational expressions

$$\frac{7}{14x} \quad \text{and} \quad \frac{7}{14+x}$$

can both be simplified by dividing each numerator and each denominator by 7.

111. When performing the division

$$\frac{7x}{x+3} \div \frac{(x+3)^2}{x-5},$$

I began by dividing the numerator and the denominator by the common factor, $x + 3$.

112. I subtracted $\frac{3x-5}{x-1}$ from $\frac{x-3}{x-1}$ and obtained a constant.

In Exercises 113–116, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

113. $\frac{x^2-25}{x-5} = x - 5$

114. The expression $\frac{-3y-6}{y+2}$ simplifies to the consecutive integer that follows -4.

115. $\frac{2x-1}{x-7} + \frac{3x-1}{x-7} - \frac{5x-2}{x-7} = 0$

116. $6 + \frac{1}{x} = \frac{7}{x}$

In Exercises 117–119, perform the indicated operations.

117. $\frac{1}{x^n-1} - \frac{1}{x^n+1} - \frac{1}{x^{2n}-1}$

118. $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x+1}\right)\left(1-\frac{1}{x+2}\right)\left(1-\frac{1}{x+3}\right)$

119. $(x-y)^{-1} + (x-y)^{-2}$

120. In one short sentence, five words or less, explain what

$$\frac{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}}$$

does to each number x.

Preview Exercises

Exercises 121–123 will help you prepare for the material covered in the next section.

121. If 6 is substituted for x in the equation

$$2(x-3) - 17 = 13 - 3(x+2),$$

is the resulting statement true or false?

122. Multiply and simplify: $12\left(\frac{x+2}{4} - \frac{x-1}{3}\right)$.

123. Evaluate

$$\frac{-b-\sqrt{b^2-4ac}}{2a}$$

for $a = 2$, $b = 9$, and $c = -5$.

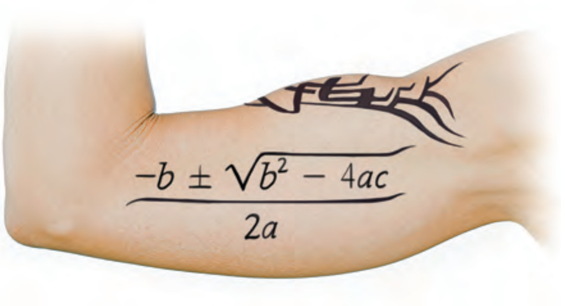
Section P.7 Equations

Objectives

1. Solve linear equations in one variable.
2. Solve linear equations containing fractions.
3. Solve rational equations with variables in the denominators.
4. Solve a formula for a variable.
5. Solve equations involving absolute value.
6. Solve quadratic equations by factoring.
7. Solve quadratic equations by the square root property.
8. Solve quadratic equations by completing the square.
9. Solve quadratic equations using the quadratic formula.
10. Use the discriminant to determine the number and type of solutions of quadratic equations.
11. Determine the most efficient method to use when solving a quadratic equation.
12. Solve radical equations.

Math tattoos. Who knew? Do you recognize the significance of this tattoo? The algebraic expression gives the solutions of a *quadratic equation*.

In this section, we will review how to solve a variety of equations, including linear equations, quadratic equations, and radical equations.



Solving Linear Equations in One Variable

We begin with a general definition of a linear equation in one variable.

Definition of a Linear Equation

A **linear equation in one variable x** is an equation that can be written in the form

$$ax + b = 0,$$

where a and b are real numbers, and $a \neq 0$.

An example of a linear equation in one variable is

$$4x + 12 = 0.$$