Naomi Chissick

Promoting Learning through Inquiry

From the very beginning of education, a child should experience the joy of discovery.

—Alfred North Whitehead

Mathematics is a subject that allows for the discovery of properties and relationships through personal inquiry (Thompson 1984). Pólya wrote

> . . . if [the teacher] challenges the curiosity of students by setting them problems proportionate to their knowledge, and helps them solve their problems with stimulating questions, [the teacher] may give them a taste for, and some means of, independent thinking. (Pólya, 1973, p. V)

In addition, *Principles and Standards for School Mathematics* stresses that all students should learn important mathematical concepts and processes with understanding (NCTM 2000, pp. 12, 20).

For the last twelve years, our efforts at ORT, which includes a network of colleges and schools for advanced technologies and sciences in Israel, have been directed toward teaching and learning mathematics through inquiry. We have adopted a three-tier process:

Reflecting on the findings is very important

Tier 1. We use short, open-ended tasks in the classroom.

Tier 2. We give longer class and homework assignments for independent learning of new material or for learning new aspects of previously learned materials.

Tier 3. We hold a competition for open-ended, inquiry-oriented, original projects in mathematics.

Each stage in the process includes group work with two or three students, brainstorming processes, discussions, and trial and error. Students benefit in multiple ways:

- Original thinking is valued.
- Teamwork is encouraged.
- Listening to peers enriches students' thinking.
- Students learn new approaches and strategies.
- Students gain questioning and hypothesizing abilities.
- Students experience methodical work.
- Reflection on findings is emphasized.
- Students learn to present findings.
- Students write mathematically.

The process is promoted mainly by seminars and discussion groups for teachers, as well with in-school support by facilitators. Teacher participation is voluntary, both in the seminars and implementation processes in the schools.

ORT places a special emphasis on including all students in the process. Exercises, worksheets, tasks, and projects are designed and used for different levels and different classes. Some are chosen from materials prepared by different educational bodies in Israel, others are adapted from articles and materials published abroad, and some are developed by the ORT team.

TIER 1: SHORT, OPEN-ENDED TASKS

Learning through discovery has been one of the cornerstones of mathematics education theory for many years. Nevertheless, many teachers are reluctant to use it, for fear of wasting time, on the one hand, and because of uncertainty about students' knowledge and ability to cope, on the other hand. Therefore, we decided to start in grades 9–10 (ages 14–16), when no external examinations loom for students in Israel, who take such examinations in eleventh and twelfth grades.

The teacher introduces the tasks gradually. Short introductory tasks are presented for classwork, while the teacher is available to answer questions and clarify dilemmas. When the students are acquainted with the general approach, the teacher presents the actual tasks.

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An example of a short, open-ended task appears in **figure 1.** For this task, most students draw two noncongruent triangles and place them on different sides of the common base, as in **figure 2.** The properties that students see immediately are the two pairs of equal angles in the isoceles triangles, with vertices *B* and *D*. The proof that $m\angle ABC = m\angle ADC$ causes no difficulties. Adding the "missing" diagonal for instruction (*d*), though, raises a few questions and reveals some misconceptions. Students assume that the heights and medians of both isosceles triangles, as well as the bisectors of the angles at *A* and *C,* lie on $\overline{AC}$; and they do not see anything to prove. The discussion that follows presents the teacher with an opportunity to clarify the difference between joining *A* and *C* by a line segment and drawing bisectors of $\angle A$ and $\angle C$ (and proving that they are both medians that meet at *E,* as well as heights, and therefore form a straight line). Accepting or showing other possibilities is important, for example, drawing two medians or heights to the common base of the triangles and completing the proof from there.

What Are the Properties of a Kite?

Two isosceles triangles with bases equal in length are given.

A quadrilateral is built by joining the bases of the triangles. This type of quadrilateral is called a *kite.*

a) Make a drawing, and label the vertices.

b) What properties of the kite can you see immediately? Write your findings in your own words.

c) Try to rephrase your findings in mathematical terms, and then work out proofs.

d) Add the missing diagonal to your drawing (if necessary, make a new drawing). What additional properties can you find?

e) Repeat step (*c*).

f) Now reflect:

- Do the properties that you describe apply to kites in general or just to your special case?
- What can you discover (and prove) about special cases?
- In what other way can you join the two isosceles triangles (using the instructions given above)?

Fig. 1
Sample short, open-ended task

The last question of the task suggests the possibility of drawing a concave kite, shown in **figure 3.** Some students draw two congruent triangles; the teacher should allow them to continue with their investigation and find the properties of the kite that they have drawn (a rhombus, as shown in **fig. 4**). They usually realize their predicament at the reflection stage.

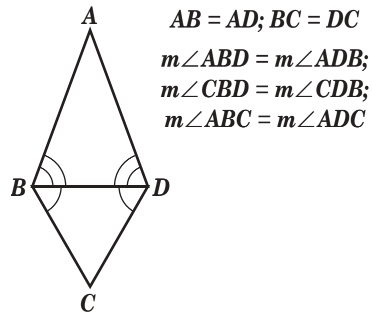


Fig. 2
Noncongruent triangles placed on different sides of a common base

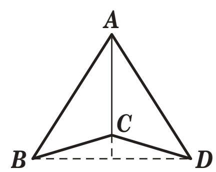


Fig. 3
A concave kite
Do the properties of the kite differ from the properties that were previously found?

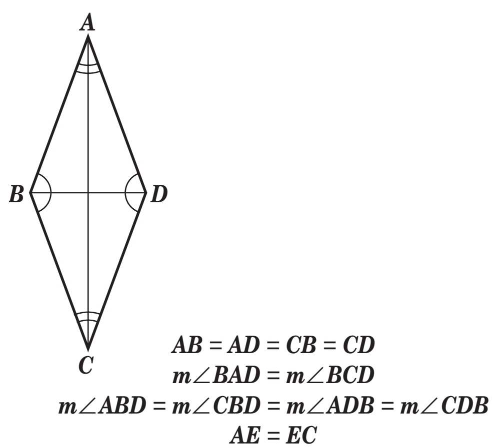


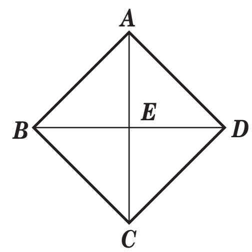
Fig. 4
Some properties added by making the kite a rhombus

Reflecting on the findings is very important and is best done through class discussion. The teacher can facilitate a discussion about the additional properties of the rhombus, the square (shown in **fig. 5**), and the concave kite. Discovering the properties of the kite through investigation fascinates students and teachers alike.

Students study a new subject, or new aspects of an old one, on their own, by inquiring and discovering, by generalizing and proving

TIER 2: ASSIGNMENTS FOR INDEPENDENT LEARNING OF NEW MATERIAL

Here we go one step further: the students are asked to study a new subject or new aspects of an old one



$AE = BE = CE = DE$
$m\angle ABC = m\angle BCD = m\angle CDA = m\angle DAB = 90$

Fig. 5
The properties added by making the kite a square

"I have rediscovered mathematics and would like to learn more of it this way"

on their own, by inquiring and discovering, generalizing and proving. Readers may think that similar activities were presented in the previous example; the three tiers are indeed different intellectual levels of the same process. The difference lies in the scope and level of the tasks rather than in their nature. The examples given here are used for precalculus and calculus students in eleventh and twelfth grades. Each of the assignments in this tier includes a worksheet with questions, exercises, and points to ponder.

Students may consult their teachers while they work on the assignments. A discussion in class should follow the process of independent learning, preferably after assessment of student learning, so that the teacher is aware of the main issues and questions arising from the task.

Power functions

Figure 6 gives an example of an independent learning assignment that leads to the discovery of even and odd power functions and their properties.

Power Functions

The functions $f(x) = x^n$; $n = 1, 2, 3, \ldots$ are called *power functions.*

a) Explain why.
b) Sketch the graphs of the power functions, where $n = 1, 2, \ldots, 8$. You may use a graphing calculator or mathematical software.
c) How can you sort these functions into two groups?
d) Suggest names for the groups that you have formed.
e) Draw the graphs of the two groups on two separate sets of axes.
f) List the properties of each group. Verify each property graphically and algebraically.
g) Generalize your findings. Give algebraic proofs or explanations in your own words.

Fig. 6
Sample independent learning assignment

After students see the graphs, shown in **figure 7,** they find that dividing the power functions into two groups is easy. By separating the graphs of the even power functions, shown in **figure 8a,** and the odd power functions, shown in **figure 8b,** they discover some of the properties and can describe them in words and algebraically (see **table 1**).

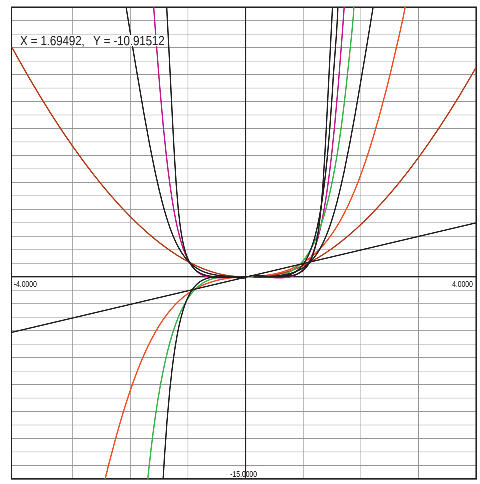


Fig. 7
Graphs of power functions

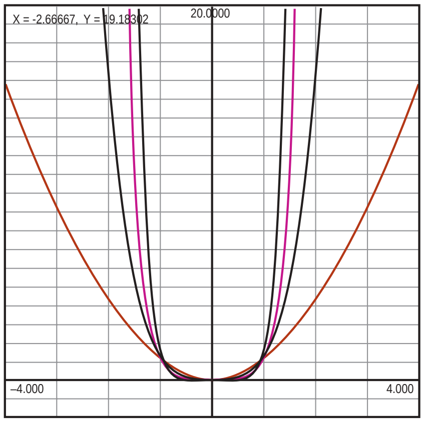
The symmetry properties of each group (especially the odd functions) are more difficult to find and to describe. Here students need more guidance, which can be given in class or on a separate worksheet. Line symmetry and point symmetry should be defined and presented both graphically and algebraically; the choice of presentation depends on the class's prior knowledge as much as on the teacher's preferences.

The generalization to other even and odd functions, for example, trigonometric functions, is then easy.

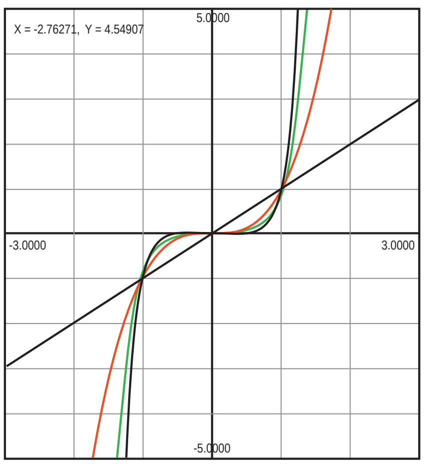
The function tan θ

This assignment, which helps students discover new concepts, was given to students in a trigonometry course. They had already found the properties of the functions sin θ and cos θ from their definitions on the unit circle. Students examined the signs and sizes of the line segments representing sin θ and cos θ in **figure 9,** as point A moves around the unit circle and $\overline{AB}$ is perpendicular to the x-axis.

The new assignment was to find the properties and draw a graph of the function tan θ in a way similar to students' previous experiences with sin θ and cos θ. Tan θ was defined as the directed length of the segment CD, which lies on the tangent to the unit circle at D, (1, 0), as shown in **figure 10.**



Even power functions
(a)



Odd power functions
(b)

Fig. 8

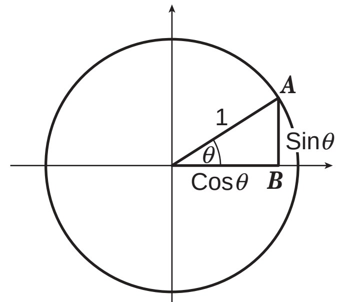


Fig. 9
Line segments representing sin θ or cos θ as point A moves around the unit circle

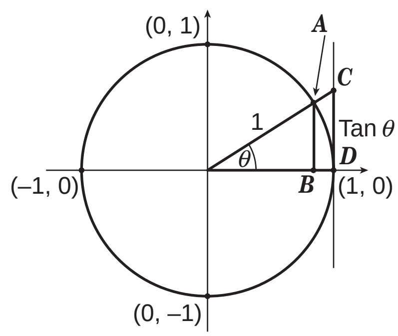


Fig. 10
Tan θ defined as $\overline{CD}$, which lies on the tangent to the unit circle at D (1, 0)

The coordinates above the x-axis are defined to be positive; the coordinates under the x-axis, negative. Almost all students performed this assignment to near perfection. The second part of the assignment consisted of an interesting inquiry: the teacher asked students to try to find a connection among tan θ, cos θ, and sin θ. Although the teacher allowed students to refer to their textbooks or other books, students' descriptions of their inquiries showed that most of them did not.

A few examples follow:

- Rami noticed that at $\theta = 45°$, tan $\theta = 1$ and sin θ = cos θ. He therefore concluded that either

Mathematical thinking is not only for a chosen few

TABLE 1

Properties of Even and Odd Families of Power Functions

Even Power Functions (fig. 8a)		Odd Power Functions (fig. 8b)	
Graphs pass through the origin	$f(0) = 0$	Graphs pass through the origin	$f(0) = 0$
Function values are nonnegative	$f(x) \geq 0$	Function values are negative for $x < 0$ and positive for $x > 0$	$x < 0 \Rightarrow f(x) < 0$ $x > 0 \Rightarrow f(x) > 0$
Function values— • decrease for $x < 0$ • increase for $x > 0$	$x < 0 \Rightarrow f'(x) < 0$ $x > 0 \Rightarrow f'(x) > 0$ (*)	Function values increase	$f'(x) > 0$
Graphs go through (–1, 1) and (1, 1)	$f(-1) = 1$ $f(1) = 1$	Graphs go through (–1, –1) and (1, 1)	$f(-1) = -1$ $f(1) = 1$

* or $x_2 < x_1 < 0 \Rightarrow f(x_2) > f(x_1)$ for decreasing;
$x_2 > x_1 > 0 \Rightarrow f(x_2) > f(x_1)$ for increasing

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

or

$$\tan\theta = \frac{\cos\theta}{\sin\theta}.$$

He then looked at acute angles greater than 45°. Realizing that $\sin\theta > \cos\theta$ for these angles and that $\tan\theta > 1$, he chose the first expression. He then checked it for angles between 0° and 45° and confirmed his result. Although his proof was not a rigorous mathematical one, Rami's work showed the elements of inquiry, starting with a special case and advancing from it, making a hypothesis and trying to prove it.

- Linda used the fact that the slope of a line is constant, so it can be calculated between any two points on the line. (See **fig. 11.**) Therefore,

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}.$$

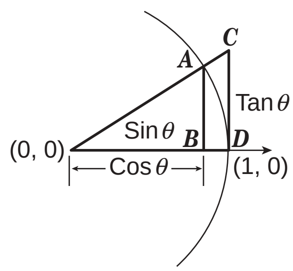


Fig. 11
Using slope to find relationships among sin *θ*, cos *θ*, and tan *θ*

- Ruth took an altogether different route. She divided the figure into sections and worked out their areas, as shown in **figure 12.**

$$\frac{\cos\theta\sin\theta}{2} + \sin\theta(1-\cos\theta) + \frac{(1-\cos\theta)(\tan\theta - \sin\theta)}{2} = \frac{(\tan\theta)\bullet 1}{2};$$

$$\begin{aligned}&\cos\theta\sin\theta + 2\sin\theta - 2\sin\theta\cos\theta + \tan\theta\\ &-\cos\theta\tan\theta - \sin\theta + \sin\theta\cos\theta\\ &\qquad = \tan\theta\ \sin\theta - \cos\theta\tan\theta\\ &\qquad = 0.\end{aligned}$$

Therefore,

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

where $\cos\theta \neq 0$.

Needless to say, I was delighted with the results. I asked students to reflect on these assignments in writing. Most of them described their apprehension when they first approached the tasks. As one of them said—

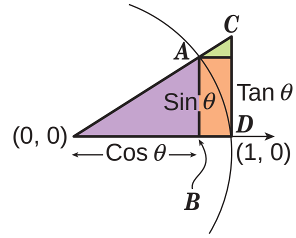


Fig. 12
Using area to find relationships among sin *θ*, cos *θ*, and tan *θ*

> At first, I was flabbergasted: by the number of pages and by the enormity of the task—to learn mathematics (of all subjects!) on my own. But as I started working on it, I became more and more fascinated and less frightened. . . . I have rediscovered mathematics and would like to learn more of it this way.

TIER 3: A COMPETITION FOR OPEN-ENDED PROJECTS IN MATHEMATICS

In 1994, we created a competition that would bring out the best and most original mathematics students in ORT. We wanted to encourage all students to participate, to reflect our belief that mathematical thinking is not only for a chosen few. And so the Competition for Open-Ended Projects in Mathematics (COPM) began.

A detailed description of the competition is beyond the scope of this article. Each project submitted for this competition must have an original contribution (for example, building an original tessellation pattern and discussing the symmetries in it) and a description of the process of inquiry. The participating students (grades 9–12) write research papers that are ten to twenty-five pages long; the authors of the best papers present their findings to the judging committee.

An interesting example of a subject for inquiry that can be a basis for an open-ended project can be found in "Equiareal Polygons: A Mathematical Conversation about a New Concept" (Winicki-Landman 2001).

The materials presented in this article, as well as similar materials developed at ORT and other educational institutions, are used by teachers who believe that introducing their students to a novel side of mathematics—one that is interesting, stimulating, and fun too—will improve the atmosphere in the classroom and make learning mathematics more enjoyable. And the students never cease to amaze us with their enthusiasm and their original ideas.

Implementing changes is not easy. It is a long, complex process and is loaded with uncertainty and risk taking (Fullan and Miles 1992). Nevertheless, many teachers are willing and able to try new ideas; together we plan, implement, reflect, change, and try again. We all believe that the students are the ones who benefit most from this process, and this belief is the fuel for our efforts.

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